Variational Bounds and Sensitivity Analysis of Traffic Processes in Continuous Flow Models

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Abstract

This paper is concerned with functional variations in processes associated with Continuous Flow Models (CFMs). The considered CFMs are characterized (defined) by their inflow-rate process, service-rate process, and buffer size, and have associated with them various derived processes such as the workload (buffer contents), outflow and overflow processes. The paper's main result is comprised of Lipchitz continuity of the mappings between the defining processes and the derived processes. This result is extended from single-flow CFMs to multi-flow CFMs, where it is used to establish the unbiasedness of IPA for workload-related and loss-related performance functions.

Key words. Continuous Flow Models, sensitivity analysis, IPA.

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1 Introduction

Continous flow models (CFM) provide an alternative modeling technique to queueing networks in various application domains, such as telecommunications and manufcturing [1, 7, 12, 14, 17]. The CFM is a "fluidized" queue whose inflow process, service process and outflow process are characterized by flow rates, and the workload process (buffer content) is defined by the fluid volume stored in the buffer. The buffer may be finite (limited) or infinite (unlimited). If the buffer is finite and full, excess fluid that cannot be absorbed is turned away and considered lost. Figure 1 shows a typical CFM whose inflow rate and service rate are denoted by $\alpha(t)$ and $\beta(t)$, respectively; c denotes the buffer size; $x(t)$ denotes the workload; $\delta(t)$ denotes the outflow rate; and $\gamma(t)$ denotes the overflow rate due to finite buffer.[1]

Our motivation is derived from high-speed, packet-switching and cell-switching telecommunications networks. Here, a cell (or packet) may be likened to a molecule in a fluid flow in that its effect on the teletraffic stream is virtually infinitesimal, thereby justifying a CFM representation. The idea of using CFM to model data traffic goes back almost two decades to [1], and [7], which employed simple CFMs to analyze multiplexed data streams. More recently, CFMs have been considered for simulation of high-speed networks [5, 8, 23], and shown to possess some advantages and some disadvantages as compared to queueing models [10].

This paper is concerned with variational bounds on the mappings between various processes associated with CFMs. Specifically, a CFM often is characterized by its inflow rate, service rate and buffer size, from which other processes, like the workload, outflow rate and overflow rate are derived. Correspondingly, we call the processes characterizing the CFM's behavior *defining processes*, and we label the processes computable from the defining processes the *derived processes*. Our primary objective here is to establish continuity, in a suitable topology, of the mappings from the defining processes to the derived processes. In particular, we compute Lipschitz bounds on these mappings, and demonstrate them to be sharp in some cases. Starting with single-flow CFMs we later extend the results to multi-flow CFMs as well.

Consequent to the above Lipschitz continuity of the interprocess mappings, we establish the unbiasedness of IPA for a number of performance measures defined in the CFM setting. In particular, we consider the loss volume and the average workload in multi-flow CFMs, chosen for their simplicity and importance in the study of telecommunications networks. We point out that, in the queueing setup, such performance measures tend to give biased IPA estimators, and although alternative perturbation-analysis techniques can be employed (like SPA), they typically require more computing efforts than IPA.

A CFM can be viewed as a dynamical system whose state is the workload process (see Section 2). Many performance measures of interest, such as loss volumes and sojourn times, can be expressed in terms of the workload trajectory. Consequently,

[1] The explicit dependence on t is omitted in Figure 1.

establishing the Lipschitz continuity of the mappings between the defining processes and the workload process can lead to unbiased IPA estimators for various performance functions. Furthermore, low-complexity SPA estimators may be obtained for performance measures with discontinuities in the terminal reward, as long as the state variables are continuous (see, e.g., [19] for examples and discussion of this point in the queueing setting). We shall not, however, pursue this issue any further in this paper, but defer it to a forthcoming publication. Our goal in this paper is just to establish the Lipschitz continuity of the inter-process mappings in single-flow CFMs and some multi-flow CFMs, and derive some applications to the unbiasedness of IPA.

IPA has been considered in the context of fluid-flow networks in manufacturing applications, and especially in transfer lines, where it was shown to give unbiased derivative estimates [12, 14, 16, 17, 22]. Extensions to continuous-flow Petri networks were made in [21]. An interesting telecommunications application to traffic throughput in an ATM switch was considered in [11]. The approach in this paper is different in that it is based on general functional variational bounds, which may be applicable to a variety of CFM networks and performance functions. Such extensions and applications may be considered in the future.

The rest of the paper is organized as follows. Section 2 presents the models and establishes notation. Section 3 considers single-flow CFMs and derives Lipschitz bounds for mappings between their defining processes and derived processes. Section 4 extends results of Section 3 to multi-flow CFMs, and applies them to establish unbiasedness of IPA for special cases. Finally, Section 5 summarizes the main results derived throughout the paper. The appendix contains proofs of propositions stated but left unproven in the main body of the paper.

2 Continuous-Flow Models (CFM)

Consider the fluid model shown in Figure 1. Here $\alpha(t)$ denotes the inflow rate, $\beta(t)$ is the service rate, and $c(t)$ denotes the buffer size, all at time t (the explicit dependence on t is ommited in the figure). The time horizon is an interval $[0, T]$ for some given fixed $T > 0$. We allow the buffer size $c(t)$ to be time dependent since it simplifies the analysis to come, as we shall later see. The processes $\{\alpha(t)\}$, $\{\beta(t)\}$ and $\{c(t)\}$ are assumed to be random processes defined on a suitable probability space $(\Omega, \mathcal{F}, P)$, where a sample point (sample path) is denoted by ω. The following will be implicitely assumed throughout the paper.

Assumption 2.1 With probability one:
(i) The sample paths of $\{\alpha(t)\}$ and $\{\beta(t)\}$ are piecewise continuously differentiable in the interval $[0, T]$.
(ii) The sample paths of $\{c(t)\}$ are continuous and piecewise continuously differentiable in the interval $[0, T]$. □

We remark that "piecewise continuously differentiable" means that realizations of the above processes have finite numbers of (possibly sample-path dependent)discontinuities in the interval $[0, T]$, and they are continuously differentiable on their continuity intervals.

The processes $\{\alpha(t)\}$, $\{\beta(t)\}$ and $\{c(t)\}$ are called the *defining processes*, since they define much of the behavior of the CFM. In a similar vein, the *derived processes* are those defined in terms of the defining processes. Accordingly, Figure 1 depicts three derived processes: the workload $x(t)$, the outflow rate $\delta(t)$, and the overflow rate $\gamma(t)$. These derived processes are defined in terms of the defining processes as follows:

$$\frac{dx(t)}{dt^+} = \begin{cases} 0, & \text{if } x(t) = 0 \text{ and } \alpha(t) \leq \beta(t) \\ \dot{c}(t), & \text{if } x(t) = c(t) \text{ and } \alpha(t) - \beta(t) \geq \dot{c}(t) \\ \alpha(t) - \beta(t), & \text{otherwise;} \end{cases} \tag{2.1}$$

$$\delta(t) = \begin{cases} \beta(t), & \text{if } x(t) > 0 \\ \alpha(t), & \text{if } x(t) = 0; \end{cases} \tag{2.2}$$

$$\gamma(t) = \begin{cases} \alpha(t) - \beta(t) - \dot{c}(t), & \text{if } x(t) = c(t) \text{ and } \alpha(t) - \beta(t) \geq \dot{c}(t) \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We observe that for the constant buffer size, we have $\dot{c}(t) = 0$, and Eqs. (2.1) and (2.3) are then reduced to simpler and more familiar special cases.

A *basic CFM* is defined by a set of defining processes, subject to Assumption 2.1, and has associated with it a set of derived processes given by Eqs. (2.1) – (2.3). To ensure a unique solution for Eq. (2.1) we impose the initial condition $x(0) = 0$. The basic CFM is a dynamical system whose state trajectory consists of the workload process $x(t)$. It is assumed to have a single-class flow. Multi-flow CFMs with various multiplexing and priority schemes can be represented by multiple interacting basic CFMs. This will be shown to simplify the analysis of multi-flow CFMs and enhance the general scope of the obtained results. The next two simple lemmas will be employed in the subsequent analysis.

Lemma 2.1 For the basic CFM, the following equation holds.

$$\int_0^T \alpha(\tau)\, d\tau = \int_0^T \delta(\tau)\, d\tau + \int_0^T \gamma(\tau)\, d\tau + x(T). \tag{2.4}$$

Proof. Follows immediately from Eqs. (2.1) – (2.3). □

Lemma 2.2 For a basic CFM, fix any $t_1 \in [0, T]$ and $t_2 \in [t_1, T]$.
(a) If $x(t) < c(t)$ for every $t \in (t_1, t_2)$, then $x(t_2) - x(t_1) \geq \int_{t_1}^{t_2} [\alpha(\tau) - \beta(\tau)]\, d\tau$.
(b) If $x(t) > 0$ for all $t \in (t_1, t_2)$, then $x(t_2) - x(t_1) \leq \int_{t_1}^{t_2} [\alpha(\tau) - \beta(\tau)]\, d\tau$.

Proof. Follows immediately from Eq. (2.1). □

Various performance measures of interest can be formulated in terms of the processes associated with the CFM. In particular, the loss volume and average workload are relevant measures in the context of telecommunications networks. The loss volume, denoted by L, is defined by

$$L = \int_0^T \gamma(\tau) d\tau,$$

and the average workload, denoted by $\bar{x}$, is given by

$$\bar{x} = \frac{1}{T} \int_0^T x(\tau)\, d\tau.$$

The concept of delay, or sojourn time, is less clear since the CFM does not have individual jobs associated with it. However, we can define the virtual delay of a "molecule" that arrived at time t by the amount of time that "molecule" would stay in the buffer. Denoting this virtual delay by $s(t)$, it is readily seen that

$$s(t) = \min\{s \geq 0 : \; x(t) = \int_t^{t+s} \beta(\tau)\, d\tau\}. \tag{2.5}$$

The average (virtual) delay can be obtained either from Eq. (2.5) or from the average workload and a variant of Little's law.[2] As we shall shortly see, the virtual delay will play a role in the analysis of multiflow CFMs.

Multi-flow CFMs often can be represented by multiple interacting basic CFMs. To illustrate, consider the two-flow CFM shown in Figure 2, where the flows are multiplexed into a shared-server-shared-buffer CFM, and then routed to their respective destinations. The inflow-rate processes are $\{\alpha_1(t)\}$ and $\{\alpha_2(t)\}$, respectively, while the common-server rate process is $\{\beta(t)\}$. The shared buffer's capacity process is $\{c(t)\}$. For the multiplexing schemes to be discussed below, we will represent this multi-flow CFM by two interacting single-flow basic CFMs, as shown in Figure 3. To this end, the two inflow-rate processes of Figure 2 are labeled *type-1 flow* and *type-2 flow*, and their respective outflow-rate processes, overflow-rate processes and workload processes are similarly denoted by $\{\delta_i(t)\}$, $\{\gamma_i(t)\}$ and $\{x_i(t)\}$, $i = 1, 2$. The CFMs shown in Figure 3 have the same inflow-rate processes $\{\alpha_i(t)\}$ as in Figure 2.

Let us consider two priority schemes often arising in networking applications: preemptive priority of one fluid type over the other, and equal priorities. In either case we define the service rate processes $\{\beta_i(t)\}$ and the buffer-size process $\{c_i(t)\}$, $i = 1, 2$, in Figure 3, in such a way that the resulting outflow, overflow and workload processes are identical to those obtained for the system shown in Figure 2. This procedure generalizes to multi-flow CFMs of more than two flows, with interacting, single-flow, basic CFMs serving as building blocks.

Preemptive priority. Suppose that type-1 fluid has preemptive priority over type-2 fluid in Figure 2. This means that, an arriving type-1 "molecule" will get ahead of any type-2 molecule in the shared buffer. In case of overflow, type-1 molecules always

[2] We thank a reviewer for pointing this to us.

push type-2 molecules out of the buffer; we remark that this scheme corresponds to some implementations of congestion control algorithms in ATM switches. Thus, the type-1 flow appropriates all the available service and buffer resources, whereas the type-2 flow is given whatever is left over. In Figure 3, this implies that

$$\begin{aligned} \beta_1(t) &= \beta(t) \\ c_1(t) &= c(t), \end{aligned}$$

while

$$\begin{aligned} \beta_2(t) &= \begin{cases} 0, & \text{if } x_1(t) > 0 \\ \beta(t) - \alpha_1(t), & \text{if } x_1(t) = 0, \end{cases} \\ c_2(t) &= c(t) - x_1(t). \end{aligned}$$

The last expression for $c_2(t)$ demonstrates the representational power gained by letting basic CFMs admit time-dependent buffer size. In this example, the time dependence is a consequence of the fact that $\beta_2(t)$ and $c_2(t)$ depend on the state of the system associated with the type-1 flow.

Equal priorities. Suppose that the two flows in Figure 2 have equal priorities, so that fluid "molecules" that arrive simultaneously from the two flow sources would also depart simultaneously from the server. In other words, this is a continuous-flow analogue of the FCFS queueing discipline. Let $x_i(t)$, $i = 1, 2$, denote the amount of type-i fluid stored in the common buffer at time t, and define $x(t) = x_1(t) + x_2(t)$ and $\alpha(t) = \alpha_1(t) + \alpha_2(t)$. Thus, at each time instant t, $x(t)$ is the total workload while $\alpha(t)$ is the combined inflow rate and $\beta(t)$ is the common service rate. To simplify the exposition, we further assume here that the common buffer capacity is a constant, $c(t) = c$, $t \in [0, T]$, for some given $c \in (0, \infty)$.

Fluid "molecules" from either flow arriving at time t will depart at time $\sigma(t)$, defined by

$$\sigma(t) = t + s(t),$$

where $s(t)$ is defined by Eq. (2.5). The outflow rates of the two fluid streams at the departure time $\sigma(t)$ are proportional to their respective inflow rates at time t. We now proceed to elaborate on this point by discussing separately the cases of an infinite buffer and a finite buffer.

Infinite buffer: $c(t) = \infty$ for all $t \in [0, T]$. If $\sigma(t) = t$ (namely, fluid "molecules" arriving at time t find an empty buffer), then necessarily $\alpha(t) \leq \beta(t)$, and these "molecules" are immediately discharged at their corresponding arrival rates, $\alpha_i(t)$. If, however, fluid "molecules" arrive at a nonempty buffer, then the server necessarily discharges fluid at full capacity during the interval $[t, \sigma(t)]$. Let $\beta_i(\tau)$ denote the outflow rate of type-i fluid at time $\tau \in [t, \sigma(t)]$. Then "molecules" that arrive at time t depart at time $\sigma(t)$ at the rates $\beta_1(\sigma(t))$ and $\beta_2(\sigma(t))$, proportional to their respective arrival rates, namely,

$$\frac{\beta_i(\sigma(t))}{\beta(\sigma(t))} = \frac{\alpha_i(t)}{\alpha(t)}, \quad i = 1, 2.$$

Thus, the two-flow CFM may be represented by the two interacting basic CFMs shown in Figure 3, where the input rates $\{\alpha_i(t)\}$ are given, the buffer sizes are $c_i(t) \equiv \infty$ by assumption, and the service rates are

$$\beta_i(\sigma(t)) \;=\; \frac{\alpha_i(t)}{\alpha(t)}\,\beta(\sigma(t)) \quad i = 1, 2. \tag{2.6}$$

Finite buffer: $c(t) = c < \infty$ for all $t \in [0, T]$. Let $\gamma_i(t)$ denote the overflow rate of type-i fluid, so that $\gamma(t) = \gamma_1(t) + \gamma_2(t)$ is the combined loss rate. Overflow can occur only when the shared buffer is full and the combined inflow rate exceeds the service rate, that is, $x(t) = c$ and $\alpha(t) > \beta(t)$. In this case, both fluid flows are subjected to overflow. Moreover, the two respective overflow rates have the same relative proportion as their arrival rates, namely,

$$\gamma_i(t) \;=\; \frac{\alpha_i(t)}{\alpha(t)}\,\gamma(t) \quad i = 1, 2. \tag{2.7}$$

Let $\tilde{\alpha}_i(t) = \alpha_i(t) - \gamma_i(t)$ denote the rate at which type-i flow is admitted to the buffer, and define $\tilde{\alpha}(t) = \tilde{\alpha}_1(t) + \tilde{\alpha}_2(t)$. ¿From Eq. (2.7) it follows that the rates at which the flows are admitted to the buffer are proportional to their arrival rates, namely,

$$\frac{\tilde{\alpha}_i(t)}{\tilde{\alpha}(t)} \;=\; \frac{\alpha_i(t)}{\alpha(t)} \quad i = 1, 2.$$

Now, the two-CFM representation of the system, previously developed for the infinite-buffer case, holds for the present finite-buffer case, provided $\alpha_i(t)$ and $\alpha(t)$ are replaced by $\tilde{\alpha}_i(t)$ and $\tilde{\alpha}(t)$, respectively, in Eq. (2.6); Eq. (2.5) remains in effect. It remains to compute $\tilde{\alpha}_i(t)$, $i = 1, 2$. To this end we employ a third basic CFM, whose inflow rate, service rate and buffer capacity are, respectively, the combined rate $\alpha(t)$, the common rate $\beta(t)$, and the common buffer capacity $c(t) = c$. The dynamics of this CFM yield the combined overflow process $\gamma(t)$, whence $\gamma_i(t)$ is derived via (2.7). Consequently, the modified inflow rates, $\tilde{\alpha}_i(t) = \alpha_i(t) - \gamma_i(t)$, are readily obtained. All in all, the representation of the two-flow CFM of Figure 2 calls for an interacting set of three basic CFMs.

CFM networks can be configured by connecting various CFMs (often with multiple flows) and specifying routing and multiplexing rules. Multi-flow CFMs arise in virtual-path routing and priority-based resource allocation. This representation by means of interacting basic CFMs can be useful in analysis, as we shall see later on in Section 4.

3 Variational Bounds

This section establishes cintinuity of the mappings between defining processes and derived processes in the basic CFM. In particular, we derive Lipschitz constants which, in some cases, will be shown to be tight. The analysis here is carried out in a deterministic setup where the various processes associated with a basic CFM are considered as functions from a given interval $[0, T]$ into R^+. This setup arises in sample-path analysis where the functions involved are considered deterministic once the sample sequence is drawn.

Consider the basic CFM whose associated processes are viewed as functions from $[0, T]$ into R^+. Similarly to Assumption 2.1, we make here the following assumption.

Assumption 3.1
(i) The functions $\alpha(\cdot)$ and $\beta(\cdot)$ are piecewise continuously differentiable in the interval $[0, T]$.
(ii) The function $c(\cdot)$ is positive valued, and continuous and piecewise twice-continuously differentiable in the interval $[0, T]$. $\Box$

By Eqs. (2.1)-(2.3), the function $x(\cdot)$ is continuous and the functions $\delta(\cdot)$ and $\gamma(\cdot)$ are piecewise continuous.

We remark that the discontinuities of the functions $\alpha(\cdot)$ and $\beta(\cdot)$ can occur at different points.

We will use the metric function spaces L^1 and L^∞ with the respective norms

$$\begin{aligned} ||z||_1 &= \int_0^T |z(\tau)|\, d\tau, \\ ||z||_\infty &= \max\{|z(t)| : \ t \in [0, T]\}. \end{aligned}$$

The unqualified norm $||z||$ will stand for either one of the above norms. The L^1 norm will be used for the piecewise-continuous functions $\alpha(\cdot)$, $\beta(\cdot)$, $\delta(\cdot)$ and $\gamma(\cdot)$, while the L^∞ norm will be used for the continuous functions $x(\cdot)$ and $c(\cdot)$.

Let $u(\cdot)$ denote generically either one of the defining processes (functions), namely $\alpha(\cdot)$, $\beta(\cdot)$ or $c(\cdot)$, and let $y(\cdot)$ denote either one of the derived processes (functions) $x(\cdot)$, $\delta(\cdot)$ or $\gamma(\cdot)$ (we will employ the notation "defining processes" or "derived processes" for these functions, as appropriate). Now let us vary one, and only one of the defining processes $u(\cdot)$ to $u(\cdot) + \Delta u(\cdot)$, while keeping the other two defining processes unchanged. This will result in variations in all three derived processes, from $y(\cdot)$ to $y(\cdot) + \Delta y(\cdot)$. The Lipschitz continuity to be established is of the form

$$||\Delta y|| \ \leq \ K\, ||\Delta u||,$$

with the appropriate norms, L^1 or L^∞, and suitable Lipschitz constants K. There are nine cases to consider: variations in the three derived processes for a variation in any one of the three defining processes. In the forthcoming we will derive these Lipschitz bounds. As a matter of notation, we refer to the CFM with the defining process $u(\cdot)$ as the *nominal CFM*, and to the CFM with the defining process $u(\cdot) + \Delta u(\cdot)$, as the *perturbed CFM*. The nominal CFM and the perturbed CFM will be denoted by CFM_N and CFM_P, respectively.

Variations in the inflow rate. Let $\Delta u(t) = \Delta\alpha(t)$, that is, the functional variation is in the inflow rate function. We assume that Assumption 3.1 holds for both CFM_N and CFM_P.

To derive the requisite Lipschitz constants, we will need the following simple lemma.

Lemma 3.1 If for every $t \in [0,T]$, $\Delta\alpha(t) \geq 0$, then
(a) $\Delta x(t) \;\geq\; 0, \quad t \in [0,T]$
(b) $\Delta\delta(t) \;\geq\; 0, \quad t \in [0,T]$
(c) $\Delta\gamma(t) \;\geq\; 0, \quad t \in [0,T]$

Proof. Follows immediately from Eqs. (2.1) – (2.3). □

The next proposition shows that all three Lipschitz constants are equalt to 1.

Proposition 3.1 The following three inequalities hold,
(a) $||\Delta x||_\infty \;\leq\; ||\Delta\alpha||_1$
(b) $||\Delta\delta||_1 \;\leq\; ||\Delta\alpha||_1$
(c) $||\Delta\gamma||_1 \;\leq\; ||\Delta\alpha||_1$

Proof. The proof consists of two steps. First, we prove the proposition under the restrictive assumption that $\Delta\alpha(t) \geq 0$ for all $t \in [0,T]$. Second, we show that actually no loss of generality occurs under this assumption.

For the first step, suppose that $\Delta\alpha(t) \geq 0,\ t \in [0,T]$. Applying Lemma 2.1 first to CFM_N and then to CFM_P, and taking differences, we obtain, for every $t \in [0,T]$,

$$\int_0^t \Delta\alpha(\tau)\,d\tau \;=\; \int_0^t \Delta\delta(\tau)\,d\tau \;+\; \int_0^t \Delta\gamma(\tau)\,d\tau \;+\; \Delta x(t). \tag{3.1}$$

Since Lemma 3.1 ensures that $\Delta x(t)$ and all the integrands in (3.1) are non-negative, we readily obtain the following three inequalities,

$$\Delta x(t) \;\leq\; \int_0^t \Delta\alpha(\tau)\,d\tau \;\;\leq\;\; \int_0^T \Delta\alpha(\tau)\,d\tau \;=\; ||\Delta\alpha||_1, \quad t \in [0,T] \tag{3.2}$$

$$\int_0^T \Delta\delta(\tau)\,d\tau \;\;\leq\;\; \int_0^T \Delta\alpha(\tau)\,d\tau \tag{3.3}$$

$$\int_0^T \Delta\gamma(\tau)\,d\tau \;\;\leq\;\; \int_0^T \Delta\alpha(\tau)\,d\tau \tag{3.4}$$

Part (a) now follows from (3.2) by taking maxima over $t \in [0,T]$, while parts (b) and (c) are equivalent to (3.3) and (3.4), respectively.

For the second step, suppose that $\Delta\alpha(t)$ is general, (i.e., the assumption $\Delta\alpha(t) \geq 0$ does not necessarily hold). Define

$$\begin{aligned} \alpha_{max}(t) &= \max\{\alpha(t), \alpha(t) + \Delta\alpha(t)\}, \\ \alpha_{min}(t) &= \min\{\alpha(t), \alpha(t) + \Delta\alpha(t)\}. \end{aligned}$$

Consider now the CFM with inflow process (function) $\alpha_{min}(\cdot)$ and the CFM with inflow process $\alpha_{max}(\cdot)$, with all other defining processes being the same as before, and denote these CFMs by CFM_{min} and CFM_{max}, respectively. Let $x_{max}(\cdot)$ and $x_{min}(\cdot)$ be the resulting workload functions in CFM_{max} and CFM_{min}, respectively. Since for all $t \in [0, T]$,

$$\alpha_{min}(t) \ \leq \ \alpha(t) \ \leq \ \alpha_{max}(t)$$

and

$$\alpha_{min}(t) \ \leq \ \alpha(t) \ + \ \Delta\alpha(t) \ \leq \ \alpha_{max}(t),$$

the first monotonicity result of Lemma 3.1 implies for all $t \in [0, T]$,

$$\begin{array}{rcl} x_{min}(t) & \leq & x(t) \ \leq \ x_{max}(t) \\ x_{min}(t) & \leq & x(t) + \Delta x(t) \ \leq \ x_{max}(t) \end{array}$$

¿From the two inequalities above we deduce

$$|\Delta x(t)| \ \leq \ x_{max}(t) - x_{min}(t), \quad t \in [0, T],$$

from which it follows, by taking maxima over $t \in [0, T]$, that

$$||\Delta x||_\infty \ \leq \ ||x_{max} - x_{min}||_\infty. \tag{3.5}$$

Since $\alpha_{max}(t) - \alpha_{min}(t) \geq 0$ for all $t \in [0, T]$, part (a) of this proposition can be applied to the processes associated with CFM_{min} and CFM_{max}, to yield,

$$||x_{max} - x_{min}||_\infty \ \leq \ ||\alpha_{max} - \alpha_{min}||_1. \tag{3.6}$$

Observe that, by definition, $\alpha_{max}(t) - \alpha_{min}(t) = |\Delta\alpha(t)|$. Consequently, Eqs. (3.5) and (3.6) imply

$$||\Delta x||_\infty \ \leq \ ||\Delta\alpha||_1,$$

completing the proof of the general case of part (a).

Parts (b) and (c) can be proved by arguments similar to those of part (a). The proofs are, therefore, omitted. □

Variations in the service rate. Let $\Delta u(\cdot) = \Delta\beta(\cdot)$, that is, the variation is in the service rate function. Correspondingly, $\beta(\cdot)$ denotes the the service rate of the nominal CFM, and $\beta(\cdot) + \Delta\beta(\cdot)$ denotes the service rate of the perturbed CFM. We assume that Assumption 3.1 is valid for both CFM_N and CFM_P. The main result here is the following.

Proposition 3.2 The following three inequalities hold,
(a) $||\Delta x||_\infty \ \leq \ ||\Delta\beta||_1$
(b) $||\Delta\delta||_1 \ \leq \ 2||\Delta\beta||_1$
(c) $||\Delta\gamma||_1 \ \leq \ 2||\Delta\beta||_1$

Proof. Given in the appendix. □

Comparing this proposition to Proposition 3.1, there are a number of apparent differences. First, the proof here is much more complicated, and therefore will only be discussed here while its technical details are relegated to the appendix. Second, we note that the Lipschitz constants in parts (b) and (c) are 2, while the analogous constants in Proposition 3.1 are both 1. To illustrate why these Lipschitz constants are at least 2, let us consider the inequality in part (b), and the following example.

Example 3.1 The CFM has infinite buffer capacity (that is, $c(t) = \infty$), and $T = 2$. Let the inflow process be defined by

$$\alpha(t) \;=\; \begin{cases} 1, & \text{for } t \in [0,1] \\ 0, & \text{for } t \in (1,2]. \end{cases}$$

Fix $\varepsilon > 0$, and let the service processes be defined by

$$\beta(t) \;=\; \begin{cases} 0, & \text{for } t \in [0,1] \\ 1, & \text{for } t \in (1,2] \end{cases} \quad \text{and} \quad \beta(t) + \Delta\beta(t) \;=\; \begin{cases} 0, & \text{for } t \in [0,1] \\ 1+\varepsilon, & \text{for } t \in (1,2] \end{cases}.$$

Observe that all of the respective processes of the two CFMs are identical in the interval $[0,1]$, and $x(1) = x(1) + \Delta x(1) = 1$. Moreover, in the interval $[1,2]$ $\Delta\beta(t) = \varepsilon$, and hence $||\Delta\beta||_1 = \varepsilon$. Next, the buffer becomes empty in CFM_N at time $t = 2$, and in CFM_P at time $t = 1 + (1+\varepsilon)^{-1}$. Consequently, by (2.2), one has

$$\delta(t) \;=\; 1, \quad t \in (1,2],$$

$$\delta(t) + \Delta\delta(t) \;=\; \begin{cases} 1+\varepsilon, & t \in (1, 1+(1+\varepsilon)^{-1}] \\ 0, & t \in (1+(1+\varepsilon)^{-1}, 2]. \end{cases}$$

It follows that $||\Delta\delta||_1 = 2\varepsilon(1+\varepsilon)^{-1}$, so that the Lipschitz constant is

$$\frac{||\Delta\delta||_1}{||\Delta\beta||_1} \;=\; 2(1+\varepsilon)^{-1}.$$

Thus, the Lipschitz constant of 2 can be approached arbitrarily closely, as ε tends to zero. □

The proof of Proposition 3.2 relies on the following lemma.

Lemma 3.2 If $\Delta\beta(t) \geq 0$ for every $t \in [0,T]$, then
(a) $\Delta x(t) \;\leq\; 0, \quad t \in [0,T]$
(b) $\Delta\gamma(t) \;\leq\; 0, \quad t \in [0,T]$

Proof. Follows immediately from Eq. (2.1) and (2.3). □

Observe that this is a monotonicity result, not dissimilar to Lemma 3.1, which was used in the proof of Proposition 3.1. However, there is one difference: whereas Lemma

3.1 has three inequalities, Lemma 3.2 has only two inequalities. Missing from Lemma 3.2 is an inequality concerning $\Delta\delta(t)$, like the second inequality in Lemma 3.1. We next explain why such an inequality does not necessarily hold. If $\Delta\beta(t) \geq 0$ for all $t \in [0, T]$, namely the server of CFM_P is faster than the server of CFM_N, then clearly there will be less fluid in the buffer and less overflow at any time t in the perturbed CFM. However, regarding the quantity $\Delta\delta(t)$, it may happen that, at some time t, the buffer is empty in CFM_P while it is nonempty in CFM_N. In that case, Eq. (2.2) implies that $\delta(t) + \Delta\delta(t) = \alpha(t)$ while $\delta(t) = \beta(t)$. The difference $\Delta\delta(t) = \alpha(t) - \beta(t)$ may be either positive or negative.

This absence of a pointwise inequality concerning $\Delta\delta(t)$ greatly complicates the analysis as compared to that of the proof of Proposition 3.1. Part (b) of Proposition 3.2 is where the problem lies; part (a) is relatively simple, and part (c) follows from parts (a) and (b). We next outline the main arguments used in the proof of part (b) (while leaving the details to the appendix). Let us assume that $\Delta\beta(t) \geq 0$, $t \in [0, T]$. It then suffices to show that the following inequality is satisfied,

$$\int_0^T |\Delta\delta(\tau)| d\tau \;\leq\; 2\int_0^T \Delta\beta(\tau) d\tau. \tag{3.7}$$

To achieve this, we will find a partition of the interval $[0, T]$ into subintervals I_k, $k = 1, \ldots, m$, such that, for every $k = 1, \ldots, m$,

$$\int_{I_k} |\Delta\delta(\tau)| d\tau \;\leq\; 2\int_{I_k} \Delta\beta(\tau) d\tau. \tag{3.8}$$

We point out that the latter inequality does not necessarily hold for every subinterval of $[0, T]$. However, it is satisfied for each busy period and each idle period of CFM_N (but not necessarily for each busy period of CFM_P!). We remark that the concepts of busy period and idle period are defined as follows.

Definition 3.1 Consider a CFM over the interval $[0, T]$.
(a) An idle period of the CFM is a maximal subinterval of $[0, T]$ where the buffer is empty.
(b) A busy period of the CFM is the closure of a maximal subinterval of $[0, T]$ where the buffer is nonempty. □

Observe that the server may be active processing fluid during an idle period.

Let us now return to the derivation of Eq. (3.8). As earlier mentioned, we will show that the inequality in (3.8) is satisfied for every subinterval of $[0, T]$ comprised of an idle period or a busy period of CFM_N. The case of an idle period is a straightforward. To see this we observe that, by the first part of Lemma 3.2, an idle period of CFM_N is a subset of an idle period of CFM_P. Hence, and by Eq. (2.2), $\delta(t) = \alpha(t)$ and $\delta(t) + \Delta\delta(t) = \alpha(t)$ throughout the idle period of CFM_N. Consequently $\Delta\delta(t) = 0$, and Eq. (3.8) is satisfied in a trivial way.

The case where I_k is a busy period of CFM_N is more challenging. The reason is that this busy period may contain busy periods of CFM_P and subsets of idle periods of CFM_P. Consider the latter case and suppose, for the sake of the example, that I_k is a busy period in CFM_N and an idle period in CFM_P. Now for all $t \in I_k$, $x(t) > 0$ while $x(t) + \Delta x(t) = 0$. By Eq. (2.2), $\delta(t) = \beta(t)$ while $\delta(t) + \Delta\delta(t) = \alpha(t)$, and hence, $\Delta\delta(t) = \alpha(t) - \beta(t)$. The problem is that the latter difference term may be positive in parts of I_k and negative in other parts of I_k. To overcome this problem, we further subdivide the interval I_k into subsets where inequalities like (3.8) are satisfied. These subsets have to be carefully chosen, and they turn out to be disconnected rather than intervals. The details are in the proof which can be found in the appendix.

Variations in the buffer size. Let $\Delta u(\cdot) = \Delta c(\cdot)$, namely, the variation is in the buffer size. Assumption 3.1 is assumed in force for both CFM_N and CFM_P.

The continuity property derived below is not as strong as Proposition 3.1 or Proposition 3.2 for the respective cases of variations in the inflow rate or the service rate. In particular, the Lipschitz constants associated with $\Delta\delta$ can be much larger here than in the previous cases, and the Lipschitz constant associated with $\Delta\gamma$ is not expressed in terms of the L^1 norm, but rather in a weaker form. These Lipschitz constants are associated with the number of busy periods the CFMs have in the interval $[0, T]$. Let K_N and K_P denote the respective numbers of busy periods of CFM_N and CFM_P in the interval $[0, T]$ during which some loss is incurred, and define $K = \max\{K_N, K_P\}$.

Proposition 3.3 The following three inequalities are satisfied.
(a) $||\Delta x||_\infty \leq ||\Delta c||_\infty$
(b) $||\Delta\delta||_1 \leq (K+1)||\Delta c||_\infty$
(c) $|\int_0^T \Delta\gamma(\tau)\, d\tau| \leq K||\Delta c||_\infty$

Proof. Can be found in the appendix. □

The Lipschitz constants in parts (b) and (c) depend on the trajectory of the state $x(\cdot)$. We later will relate this Lipschitz constant to a quantity that is expressed in terms of the defining processes.

To see that the Lipschitz constant K can be sharp, consider the following example pertaining to part (c) of the proposition, and shown in Figure 4.

Example 3.2 c and $c + \Delta c$ have constant values, namely $c(t) = c$ and $\Delta c(t) = \Delta c$ for all $t \in [0, T]$. The buffer becomes full at time t_1 in CFM_N, and at time t_2 in CFM_P. In both CFMs, the buffer ceases to be full at the same time, t_3, due to a change in the input rate (not shown in the figure). Now by Eq. (2.3),

$$\int_0^T \Delta\gamma(\tau) d\tau \;=\; \int_{t_1}^{t_2} \Delta\gamma(\tau) d\tau \;=\; -\int_{t_1}^{t_2} (\alpha(\tau) - \beta(\tau)) d\tau. \tag{3.9}$$

On the other hand, applying Eq. (2.1) to CFM_P, we obtain that

$$\int_{t_1}^{t_2} (\alpha(\tau) - \beta(\tau)) d\tau \;=\; x(t_2) + \Delta x(t_2) \;-\; (x(t_1) + \Delta x(t_1)). \tag{3.10}$$

Noting (Figure 4) that $x(t_2) + \Delta x(t_2) = c + \Delta c$ and $x(t_1) + \Delta x(t_1) = c$, we get, from (3.9) and (3.10), that

$$\int_0^T \gamma(\tau) d\tau \;=\; -\Delta c. \tag{3.11}$$

We see that the last part of Proposition 3.3 is satisfied with $K = 1$. Moreover, the inequality in Eq. (3.11) does not depend on the slopes of the defining processes, and therefore it is independent of the quantity $t_2 - t_1$, which may be as small as desired (for the sake of the example). In particular, for a given positive integer K, we can have K "peaks" as shown in Figure 5. In this case, it is seen that part (c) of Proposition 3.3 is satisfied, and the indicated Lipschitz constant K is indeed sharp.

4 Applications to IPA

This section extends the results derived in Section 3 to multi-flow CFMs, and applies them to establish the unbiasedness of IPA for the average workload and loss volume performance functions. Flow multiplexing schemes in the CFM include pre-emptive priorities as well as equal priorities, as defined in Section 2. The main results concern Lipschitz continuity of the attendant sample performance functions, from which the unbiasedness of IPA directly follows under suitable assumptions on the underlying probability space.

The discussion below will be carried out in the following IPA setting. Let $\Theta \subset \mathbb{R}$ be an open bounded interval, and let $L(\theta) : \bar{\Theta} \to \mathbb{R}$ be a real-valued random function defined over a common probability space, $(\Omega, \mathcal{F}, P)$ ($\bar{\Theta}$ denotes the closure of Θ). $L(\theta)$ is called the *sample performance function.* Its expected value as function,

$$\ell(\theta) \;=\; E[L(\theta)],$$

is assumed to be well defined and finite valued. In the context of DEDS, for a given fixed $\theta \in \Theta$, IPA computes the sample derivative $L^{'}(\theta)$, which is said to be *unbiased* if

$$E[L'(\theta)] \;=\; \ell'(\theta)$$

[3, 4][3]. Various sufficient conditions for the unbiasedness of IPA have been derived. Here, we use the formulation set forth in [13], Lemma A2, p.70, reproduced below.

Condition 4.1 (Sufficient unbiasedness conditions). Suppose the following two conditions hold:
(i) For every $\theta \in \Theta$, the sample derivative $L^{'}(\theta)$ exists w.p.1.
(ii) The random function $L(\theta)$ has a Lipschitz constant with a finite first moment.[4]
Then, the IPA derivative, $L^{'}(\theta)$, is unbiased. □

[3]It is pointed out that only the unbiasedness, and not the strong consistency, is considered in this paper. The reason is that we do not wish to be sidetracked by the question of the existence of stationary distributions.

[4]Here and elswhere, stating that a function has a Lipschitz constant means that it has a global Lipschitz constant throughout the interval $\bar{\Theta}$.

To see why the above two conditions jointly suffice for the unbiasedness of IPA, observe that, for a given fixed $\theta \in \Theta$, the following limit is satisfied w.p.1,

$$\lim_{\Delta\theta \to 0} \frac{L(\theta + \Delta\theta) - L(\theta)}{\Delta\theta} \;=\; L^{'}(\theta).$$

The Lebesgue dominated donvergence theorem, justified by part (ii) of Condition 4.1, permits the interchangeability of expectation and the above limit, which essentially constitutes the unbiasedness of IPA. For further details, please see [13].

The parameter θ and the random element $\omega \in \Omega$ can assume several forms. For instance, any one of the defining processes $u(t)$ can be a (random) function of θ. To emphasize the dependence on θ, we shall henceforth employ the notation $u(\theta; t)$, and will similarly write $y(\theta; t)$ for derived (stochastic) processes. We will assume, implicitly, that a sample path $\omega \in \Omega$ corresponds to realizations of the family of processes $\{u(\theta; t)\}_{t \in [0,T]}$ parametrized by $\theta \in \Theta$. This assumption is satisfied in a broad class of processes, including the important case of piecewise constant random functions of θ. The following example illustrates this point.

Example 4.1. The defining process is the inflow rate process $\alpha(\theta; t)$, assumed to represent a fluid source alternating between an "off" state and an "on" state. In the "off" state the inflow rate is 0, and in the "on" state the inflow rate is 1. The durations of the "off" state and the "on" state have θ-dependent respective distributions, whose realizations are denoted by $\tau(\theta)$ and $\sigma(\theta)$, respectively. Suppose that the time horizon $[0, T]$ is fixed, and assume that at time $t = 0$ the source is in the "off" state. We further assume that the number of switchover points between the "off" state and the "on" state is bounded from above by a random number M that is independent of $\theta \in \Theta$ and having a finite second moment. For example, let $\Theta = (\theta_1, \theta_2)$ where $\theta_1 > 0$, and $\tau(\theta)$ and $\sigma(\theta)$ are exponentially distributed with mean θ^{-1}. Now the sample path ω constitutes of the sequence of samples required to construct the alternating "off" and "on" periods. Corresponding to such a sample path, for a given $\theta \in \Theta$ we can construct the alternating "off" and "on" periods until the final time T is reached, and this corresponds to a realization of the process $\alpha(\theta; t)$. $\square$

More generally, it is possible to construct examples where the defining process $u(\theta; t)$ is piecewise constant, and both its jump points and constant values between jump points are random functions of θ.

Throughout this section we will consider pairs of CFMs: a nominal CFM with defining processes $u(\theta; t)$ and derived processes $y(\theta; t)$, and a perturbed CFM with defining processes $u(\theta + \Delta\theta; t)$ and derived processes $y(\theta + \Delta\theta; t)$. The corresponding CFMs will be denoted by CFM_θ and $CFM_{\theta+\Delta\theta}$, respectively. The associated difference processes will be denoted by

$$\Delta u(t) = u(\theta + \Delta\theta; t) - u(\theta; t),$$

$$\Delta y(t) = y(\theta + \Delta\theta; t) - y(\theta; t).$$

We will assume throughout the following discussion that the processes $u(\theta; t)$ and $u(\theta + \Delta\theta; t)$, and hence the difference process $\Delta u(t)$, have realizations based on a common sample path $\omega \in \Omega$. Such a common sample path will always be assumed for the difference processes.

We next turn to specific examples of the application of the variational bounds derived in Section 3 to proving the unbiasedness of IPA for several performance functions in single-flow and multi-flow CFMs.

Workload processes in multi-flow CFMs. Consider the two-flow CFM described in Section 2 and its representation as a set of two interacting basic CFMs. Two priority schemes will be examined: preemptive priority and equal priorities.

Given a bounded open interval Θ. Let $\theta \in \Theta$ be a parameter of the type-1 inflow-rate process, $\{\alpha_1(\theta;t)\}$, and let the performance function be the average type-2 workload, namely,

$$L(\theta) \;=\; \frac{1}{T}\int_0^T x_2(\theta;\tau)\,d\tau. \tag{4.1}$$

The defining processes are the inflow processes $\{\alpha_1(\theta;t)\}$ and $\{\alpha_2(t)\}$, and the common-resource processes $\{\beta(t)\}$ and $\{c(t)\}$. The following assumption is made.

Assumption 4.1 (i) With probability 1, for every $\theta \in \Theta$, the processes $\{\alpha_1(\theta;t)\}$, $\{\alpha_2(t)\}$, $\{\beta(t)\}$ and $\{\dot{c}(t)\}$ are piecewise continuously differentiable with finite numbers of jump points in $[0,T]$.
(ii) There exists a Lipschitz constant $K > 0$ with finite first moment, such that for all $\theta \in \Theta$ and $\theta + \Delta\theta \in \Theta$, $||\Delta\alpha_1||_1 \;\le\; K\,|\Delta\theta|$. □

We remark that part (ii) Assumption 4.1 is readily satisfied for the processes defined in Example 4.1. With $\alpha_1(\theta;t)$ having the form of that process, it can be seen that, for every $\theta \in \Theta$ and $\theta + \Delta\theta \in \Theta$,

$$||\Delta\alpha_1||_1 \;\le\; \frac{M^2+M}{2}|\Delta\theta|.$$

Thus, part (ii) of Asuumption 4.1 is satisfied with $K = 2^{-1}(M^2+M)$, which was assumed (in Example 4.1) to have a finite first moment.

We next consider the two service disciplines corresponding to the two priority schemes discussed in Section 2.

Preemptive priorities. Suppose that type-1 fluid has preemptive priority over type-2 fluid in the manner described in Section 2. Referring to the equivalent representation via interacting basic CFMs in Figure 3, we observe that only $\{\alpha_1(\theta;t)\}$, $\{\alpha_2(t)\}$, $\{\beta_1(t) = \beta(t)\}$ and $\{c_1(t) = c(t)\}$ are defining processes, while $\{\beta_2(\theta;t)\}$ and $\{c_2(\theta;t)\}$ are derived processes, given by

$$\beta_2(\theta;t) \;=\; \begin{cases} 0, & \text{if } x_1(\theta;t) > 0 \\ \beta(t) - \alpha_1(\theta;t), & \text{if } x_1(\theta;t) = 0 \end{cases} \tag{4.2}$$

$$c_2(\theta;t) \;=\; c(t) \;-\; x_1(\theta;t). \tag{4.3}$$

The following preliminary result will be useful in the derivation of the Lipschitz constants.

Lemma 4.1 If $\Delta\alpha_1(t) \geq 0$ for all $t \in [0, T]$, then for any $0 \leq u \leq v \leq T$,

$$\Delta x_1(v) - \Delta x_1(u) \ \leq \ \int_u^v \Delta\alpha_1(\tau)\, d\tau. \tag{4.4}$$

Proof. Analogously to Eq. (2.4) in Lemma 2.1, we have the fluid conservation equation

$$x_1(\theta, v) \ - \ x_1(\theta; u) \ = \ \int_u^v \alpha_1(\theta; \tau)\, d\tau \ - \ \int_u^v \delta_1(\theta; \tau)\, d\tau \ - \ \int_u^v \gamma_1(\theta; \tau)\, d\tau. \tag{4.5}$$

Since (4.5) holds also for $CFM_{\theta+\Delta\theta}$, differencing yields

$$\Delta x_1(v) \ - \ \Delta x_1(u) \ = \ \int_u^v \Delta\alpha_1(\tau)\, d\tau \ - \ \int_u^v \Delta\delta_1(\tau)\, d\tau \ - \ \int_u^v \Delta\gamma_1(\tau)\, d\tau.$$

Eq. (4.4) now readily follows from Lemma 3.1, which ensures that $\int_u^v \Delta\delta_1(\tau)\, d\tau \geq 0$ and $\int_u^v \Delta\gamma_1(\tau)\, d\tau \geq 0$. □

Proposition 4.1 The sample-performance function $L(\theta)$ in Eq. (4.1) is Lipschitz continuous, and the Lipschitz constant is $2K$, where K is given by Assumption 4.1.

Proof. The proof relies on the representation of the two-flow CFM in Figure 2 as two interacting basic CFMs. To this end, fix $\theta \in \Theta$ and $\theta + \Delta\theta \in \Theta$. We first show that

$$||\Delta\beta_2||_1 \ \leq \ ||\Delta\alpha_1||_1. \tag{4.6}$$

If $\Delta\alpha_1(t) \geq 0$ for all $t \in [0, T]$ then $\Delta x_1(t) \geq 0$ for all $t \in [0, T]$ (by Lemma 3.1), and consequently $\Delta\beta_2(t) \leq 0$ and $\Delta c_2(t) \leq 0$ for all $t \in [0, T]$ (from Eqs. (4.2) and (4.3), respectively). Therefore, in order to prove Eq. (4.6), it suffices to assume, without loss of generality, that $\Delta\alpha_1(t) \geq 0$ for all $t \in [0, T]$, and then prove the inequality

$$\int_0^T \Delta\beta_2(\tau)\, d\tau \ \geq \ -\int_0^T \Delta\alpha_1(\tau)\, d\tau. \tag{4.7}$$

Suppose then that $\Delta\alpha_1(t) \geq 0$, $t \in [0, T]$. By Assumption 4.1 and the implied condition $\Delta x_1(t) \geq 0$, $t \in [0, T]$, there exists a partition $[0, T) = \cup_{j=1}^m [t_j, t_{j+1})$ of the interval $[0, T)$, such that for every $j = 1, \ldots, m$, exactly one of the following three conditions holds for all $t \in (t_j, t_{j+1})$:
(i) $x_1(\theta + \Delta\theta; t) = 0$
(ii) $x_1(\theta + \Delta\theta; t) > 0$ and $x_1(\theta; t) = 0$
(iii) $x_1(\theta; t) > 0$
In either case we next prove the inequality

$$\int_{t_j}^{t_{j+1}} \Delta\beta_2(\tau)\, d\tau \ \geq \ \Delta x_1(t_{j+1}) \ - \ \Delta x_1(t_j) \ - \ \int_{t_j}^{t_{j+1}} \Delta\alpha_1(\tau)\, d\tau. \tag{4.8}$$

In Case (i), $\Delta x_1(t_j) = \Delta x_1(t_{j+1}) = 0$ by the implied condition $\Delta x_1(t) \geq 0$, $t \in [0, T]$, and by Eq. (4.2),

$$\int_{t_j}^{t_{j+1}} \Delta\beta_2(\tau)\, d\tau \;=\; -\int_{t_j}^{t_{j+1}} \Delta\alpha_1(\tau)\, d\tau.$$

Hence, (4.8) holds with an equality.
In Case (ii), $x_1(\theta; t_j) = x_1(\theta; t_{j+1}) = 0$, so that

$$\Delta x_1(t_j) \;=\; x_1(\theta + \Delta\theta; t_j), \quad \Delta x_1(t_{j+1}) \;=\; x_1(\theta + \Delta\theta; t_{j+1}). \tag{4.9}$$

Moreover, by Eq. (4.2) and the conditions of Case (ii),

$$\Delta\beta_2(t) \;=\; -\beta_2(\theta; t) \;=\; \alpha_1(\theta; t) \;-\; \beta(t), \quad t \in (t_j, t_{j+1}),$$

which on subtracting and adding $\alpha_1(\theta + \Delta\theta; t)$ yields

$$\Delta\beta_2(t) \;=\; -\Delta\alpha_1(t) \;+\; \alpha_1(\theta + \Delta\theta; t) \;-\; \beta(t).$$

Next, apply part (b) of Lemma 2.2 to the type-1 flow in $CFM_{\theta+\Delta\theta}$ over the time interval $[t_j, t_{j+1})$. Recalling that $\beta_1(t) = \beta(t)$, we obtain,

$$\int_{t_j}^{t_{j+1}} \Delta\beta_2(\tau)\, d\tau \;\geq\; -\int_{t_j}^{t_{j+1}} \Delta\alpha_1(\tau)\, d\tau \;+\; x_1(\theta + \Delta\theta; t_{j+1}) \;-\; x_1(\theta + \Delta\theta; t_j). \tag{4.10}$$

Eq. (4.8) now follows from Eqs. (4.10) and (4.9).
Finally, in Case (iii), the associated condition and the implied condition $\Delta x_1(t) \geq 0$, $t \in [0, T]$ together imply that $x_1(\theta + \Delta\theta; t) > 0$ for all $t \in (t_j, t_{j+1})$. Hence, from Eq. (4.2), $\Delta\beta_2(t) = 0$. Applying Lemma 4.1 for $u = t_j$ and $v = t_{j+1}$ yields

$$\Delta x_1(t_{j+1}) \;-\; \Delta x_1(t_j) \;\leq\; \int_{t_j}^{t_{j+1}} \Delta\alpha_1(\tau)\, d\tau.$$

But since $\Delta\beta_2(t) = 0$, the equation above implies Eq. (4.8).

Having proved Eq. (4.8) for all three cases, we next proceed to sum it over $j = 1, \ldots, m$ and obtain

$$\int_0^T \Delta\beta_2(\tau)\, d\tau \;\geq\; \Delta x_1(T) \;-\; \Delta x_1(0) \;-\; \int_0^T \Delta\alpha_1(\tau)\, d\tau.$$

Recall that $\Delta x_1(T) \geq 0$ by the implied condition $\Delta x_1(t) \geq 0$, $t \in [0, T]$, while $\Delta x_1(0) = 0$, so the inequality above implies the inequality (4.7). This, in turn, was mentioned to imply the inequality (4.6).

Consider next the process $\{\Delta c_2(t)\}$. By Eq. (4.3), $\Delta c_2(t) = -\Delta x_1(t)$, while by Proposition 3.1, $||\Delta x_1||_\infty \leq ||\Delta\alpha_1||_1$. Consequently,

$$||\Delta c_2||_\infty \;\leq\; ||\Delta\alpha_1||_1. \tag{4.11}$$

We can now conclude the following chain of inequalities

$$||\Delta x_2||_\infty \;\leq\; ||\Delta\beta_2||_1 \;+\; ||\Delta c_2||_\infty \;\leq\; 2\, ||\Delta\alpha_1||_1 \;\leq\; 2\, K |\Delta\theta|,$$

where the first inequality follows from the applications of Proposition 3.2 and Proposition 3.3 to the type-2 flow processes in CFM_θ and $CFM_{\theta+\Delta\theta}$, the second inequality is a direct consequence of Eqs. (4.6) and (4.11), and the third inequality is just part (ii) of Assumption 4.1, which stipulates the existence of the random Lipschitz constant K. Eq. (4.1) now yields the inequality

$$|\Delta L| \;=\; |L(\theta+\Delta\theta) \;-\; L(\theta)| \;\leq\; 2K\,|\Delta\theta|,$$

thereby completing the proof. □

Corollary 4.1 If for every $\theta \in \Theta$ the sample derivative $L^{'}(\theta)$ exists with probability 1, then this IPA derivative is unbiased.

Proof. $L(\theta)$ is Lipschitz continuous and the Lipschitz constant has a finite first moment by part (ii) of Assumption 4.1. The result now follows from Condition 4.1. □

Equal priorities. We consider here only the case of infinite buffer, since it greatly simplifies the analysis; however, Assumption 4.1 still remains in effect. Accordingly, the defining processes are the inflow processes $\{\alpha_1(\theta;t)\}$ and $\{\alpha_2(t)\}$, and the common service process $\{\beta(t)\}$; and by assumption, $c(t) \equiv \infty$, so no loss can ever occur. As before, θ is a parameter of the type-1 inflow rate, and the performance function considered is the average type-2 workload, given in Eq. (4.1).

Following the notation adopted in Section 2, we define $\alpha(\theta;t) = \alpha_1(\theta;t) + \alpha_2(t)$, $x(\theta;t) = x_1(\theta;t) + x_2(\theta;t)$, and $\delta(\theta;t) = \delta_1(\theta;t) + \delta_2(\theta;t)$, and note that $\beta(t) = \beta_1(\theta;t) + \beta_2(\theta;t)$. Of course $\gamma_i(\theta;t) = \gamma_i(t) = 0$, $t \in [0,T]$. A "molecule" that enters the buffer at time t would experience a sojourn time of

$$s(\theta;t) \;=\; \min\{s \geq 0:\; x(\theta;t) = \int_t^{t+s} \beta(\theta;\tau)\,d\tau\}, \tag{4.12}$$

and from Eq. (2.6),

$$\beta_i(\theta;t+s(\theta;t)) \;=\; \frac{\alpha_i(\theta;t)}{\alpha(\theta;t)}\,\beta(t+s(t)), \quad i=1,2. \tag{4.13}$$

The following two results will be useful in establishing the Lipschitz continuity.

Lemma 4.2 If $\Delta\alpha_1(t) \geq 0$ for all $t \in [0,T]$, then
(a) $\int_0^T \Delta\beta_1(\tau)\,d\tau \geq 0$
(b) $\int_0^T \Delta\delta_1(\tau)\,d\tau \geq 0$
(c) $\int_0^T \Delta\beta_2(\tau)\,d\tau \leq 0$
(d) $\int_0^T \Delta\delta_2(\tau)\,d\tau \leq 0$

Proof. Follows from Eq. (4.13). □

The inequalities $\Delta x_1(T) \geq 0$ and $\Delta x_2(T) \geq 0$ are less obvious and require proof as follows.

Lemma 4.3 If $\Delta\alpha_1(t) \geq 0$ for all $t \in [0, T]$, then
(a) $\Delta x_1(T) \geq 0$
(b) $\Delta x_2(T) \geq 0$

Proof. Consider first $\Delta x_2(T)$. Applying Lemma 2.1 to the type-2 processes in CFM_θ and $CFM_{\theta+\Delta\theta}$ and taking differences, we obtain

$$\int_0^T \Delta\delta_2(\tau)\, d\tau \;+\; \Delta x_2(T) \;=\; 0. \tag{4.14}$$

Since $\int_0^T \Delta\delta_2(\tau)\, d\tau \leq 0$ by Lemma 4.2(d), it follows that $\Delta x_2(T) \geq 0$.

Consider next $\Delta x_1(T)$. Applying Lemma 2.1 to the type-1 processes in CFM_θ and $CFM_{\theta+\Delta\theta}$ and taking differences, we obtain

$$\int_0^T \Delta\delta_1(\tau)\, d\tau \;+\; \Delta x_1(T) \;=\; \int_0^T \Delta\alpha_1(\tau)\, d\tau. \tag{4.15}$$

By assumption, $\int_0^T \Delta\alpha_1(\tau)\, d\tau \geq 0$, and by Lemma 4.2(b), $\int_0^T \Delta\delta_1(\tau)\, d\tau \geq 0$. Thus, to conclude that $\Delta x_1(T) \geq 0$, we only need to show that

$$\int_0^T \Delta\delta_1(\tau)\, d\tau \;\leq\; \int_0^T \Delta\alpha_1(\tau)\, d\tau. \tag{4.16}$$

To this end, consider $\sigma(\theta; t) = t + s(\theta; t)$ (the "departure time" of "molecules" arriving at time t) as a function of t for fixed $\theta \in \Theta$. We are interested in the inverse function, $\sigma^{-1}(\theta; r)$, namely, the "arrival time" of fluid "molecules" (of either type) that "depart" at time r. Strictly speaking, $\sigma^{-1}(\theta; r)$ is not well-defined at points r such that $\delta(\theta; r) = 0$, since no output takes place at time r; however, in this case, we merely define $\sigma^{-1}(\theta; r) = r$. Observe that

$$\int_0^T \delta_i(\theta; \tau)\, d\tau \;=\; \int_0^{\sigma^{-1}(\theta;T)} \alpha_i(\theta; \tau)\, d\tau, \quad i = 1, 2, \tag{4.17}$$

since fluid "molecules" can "depart" on or before time T if and only if they "arrive" on or before time $\sigma^{-1}(\theta; T)$. Applying Eq. (4.17) to the type-2 fluid in CFM_θ and $CFM_{\theta+\Delta\theta}$, noting that $\alpha_2(t)$ does not depend on θ and taking differences, we get

$$0 \;\geq\; \int_0^T \Delta\delta_2(\tau)\, d\tau \;=\; \int_0^{\sigma^{-1}(\theta+\Delta\theta;T)} \alpha_2(\tau)\, d\tau \;-\; \int_0^{\sigma^{-1}(\theta;T)} \alpha_2(\tau)\, d\tau, \tag{4.18}$$

where the inequality is due to part (d) of Lemma 4.2. Since $\alpha_2(t) \geq 0$ for all t, it follows from the inequality in (4.18) that $\sigma^{-1}(\theta + \Delta\theta; T) \leq \sigma^{-1}(\theta; T)$. Consequently, an application of Eq. (4.17) to the type-1 processes in CFM_θ and $CFM_{\theta+\Delta\theta}$, followed by differencing, yields

$$\int_0^T \Delta\delta_1(\tau)\, d\tau \;=\; \int_0^{\sigma^{-1}(\theta+\Delta\theta;T)} \Delta\alpha_1(\tau)\, d\tau \;-\; \int_{\sigma^{-1}(\theta+\Delta\theta;T)}^{\sigma^{-1}(\theta;T)} \alpha_1(\theta; \tau)\, d\tau. \tag{4.19}$$

Eq. (4.16) now follows from Eq. (4.19) by observing that the last integral in (4.19) is non-negative; that $\Delta\alpha_1(t) \geq 0$ for all t by the lemma's assumption; and that $\sigma^{-1}(\theta + \Delta\theta; T) \leq T$. Finally, the inequality $\Delta x_1(T) \geq 0$ was mentioned to follow from Eq. (4.16), completing the proof of the lemma. □

Proposition 4.2 The sample-performance function $L(\theta)$ is Lipschitz continuous, and the Lipschitz constant is K, given in part (ii) of Assumption 4.1.

Proof. Fix $\theta \in \Theta$ and $\theta + \Delta\theta \in \Theta$. In view of Eq. (4.1) it suffices to establish the following inequality,

$$||\Delta x_2||_\infty \;\leq\; ||\Delta\alpha_1||_1. \tag{4.20}$$

To prove Eq. (4.20), it suffices to assume, without loss of generality, that $\Delta\alpha_1(t) \geq 0$ for all $t \in [0, T]$. Now, Proposition 3.1 implies

$$|\Delta x(T)| \;\leq\; ||\Delta\alpha||_1. \tag{4.21}$$

Note that the lefthand side of (4.21) can be rewritten as

$$\Delta x_1(T) + \Delta x_2(T) \;=\; \Delta x(T) \;=\; |\Delta x(T)|,$$

because from Lemma 4.3, $\Delta x_1(T) \geq 0$ and $\Delta x_2(T) \geq 0$. Noting further that $\Delta\alpha(t) = \Delta\alpha_1(t)$ we can rewrite (4.21) as

$$\Delta x_1(T) \;+\; \Delta x_2(T) \;\leq\; \int_0^T \Delta\alpha_1(\tau)\, d\tau,$$

from which it follows that

$$\Delta x_2(T) \;\leq\; \int_0^T \Delta\alpha_1(\tau) d\tau$$

by part (a) of Lemma 4.3. Replacing T by t in the latter inequality and noting that $\Delta\alpha_1(t) \geq 0$, we obtain, for all $t \in [0, T]$,

$$\Delta x_2(t) \;\leq\; \int_0^t \Delta\alpha_1(\tau)\, d\tau \;\leq\; \int_0^T \Delta\alpha_1(\tau)\, d\tau \;=\; ||\Delta\alpha_1||_1, \quad t \in [0, T]. \tag{4.22}$$

Eq. (4.20) now follows on maximizing the lefthand side above over $t \in [0, T]$. $\square$

Corollary 4.2 If for every $\theta \in \Theta$ the sample derivative $L'(\theta)$ exists with probability 1, then this IPA derivative is unbiased.

Proof. $L(\theta)$ is Lipschitz continuous and the Lipschitz constant has a finite first moment by part (ii) of Assumption 4.1. The result now follows from Condition 4.1. $\square$

Loss volume. The system discussed here is the basic CFM, having a single flow class. Extensions to multi-flow systems may be possible in view of the equivalent representations as outlined in Section 2.

Let us consider the loss volume performance measure as a function of parameters of the inflow rate first, and the buffer size, second. In either case the sample performance function is given by

$$L(\theta) \;=\; \int_0^T \gamma(\theta; \tau)\, d\tau. \tag{4.23}$$

Derivative with respect to inflow-rate parameters. Let the defining processes be $\{\alpha(\theta;t)\}$, $\{\beta(t)\}$ and $\{c(t)\}$, and suppose that the following assumption is in force.

Assumption 4.2
(i) With probability 1, for every $\theta \in \Theta$, the processes $\{\alpha(\theta;t)\}$, $\{\beta(t)\}$ and $\{\dot{c}(t)\}$ are piecewise continuously differentiable with finite numbers of jump points in $[0,T]$.
(ii) There exists Lipschitz constant $K > 0$ with a finite first moment, such that for all $\theta \in \Theta$ and $\theta + \Delta\theta \in \Theta$, $||\Delta\alpha||_1 \leq K|\Delta\theta|$. $\square$

Proposition 4.3 If for every $\theta \in \Theta$ the sample derivative $L^{'}(\theta)$ exists with probability 1, then this IPA derivative is unbiased.

Proof. For given $\theta \in \Theta$ and $\theta+\Delta\theta \in \Theta$, we observe that $|L(\theta+\Delta\theta)-L(\theta)| \leq ||\Delta\gamma||_1$. By Proposition 3.1 $||\Delta\gamma||_1 \leq ||\Delta\alpha||_1$, and by Assumption 4.2, $||\Delta\alpha_1|| \leq K|\Delta\theta|$. Consequently, we have that

$$|L(\theta + \Delta\theta) \ - \ L(\theta)| \ \leq \ K|\Delta\theta|,$$

and the result now follows from Condition 4.1 in conjunction with Assumption 4.2. $\square$

Derivative with respect to the buffer size. Let the variable parameter be the buffer size, namely $c(\theta;t) = \theta$ for all $t \in [0,T]$ and $\theta \in \Theta$. The other two defining processes, $\{\alpha(t)\}$ and $\{\beta(t)\}$, are assumed independent of θ. Suppose that the following assumption is in force.

Assumption 4.3
(i) The processes $\{\alpha(t)\}$ and $\{\beta(t)\}$ are piecewise continuously differentiable.
(ii) The numbers of jump points of the processes $\{\alpha(t)\}$ and $\{\beta(t)\}$ in the interval $[0,T]$ have finite first moments.
(iii) The numbers of points at which the slope processes $\{\dot{\alpha}(t)\}$ and $\{\dot{\beta}(t)\}$ change signs in the interval $[0,T]$ have finite first moments.
(iv) The lefthand point of the interval Θ is positive. $\square$

Proposition 4.4 If for every $\theta \in \Theta$ the sample derivative $L^{'}(\theta)$ exists with probability 1, then this IPA derivative is unbiased.

Proof. By Assumption 4.3, there exists a random number $\bar{K} > 0$ having a finite first moment such that, for all $\theta \in \Theta$, the number of busy periods of the CFM is bounded from above by $\bar{K}$. Next we observe that, for all $\theta \in \Theta$ and $\theta + \Delta\theta \in \Theta$,

$$|L(\theta + \Delta\theta) - \text{Ł}(\theta)| \ = \ |\int_0^T \gamma(\theta;\tau)d\tau|,$$

and, $||\Delta c||_\infty = |\Delta\theta|$. Let K be the Lipschitz constant given in part (c) of Proposition 3.3, and we observe that $K \leq \bar{K}$. This proves that $\bar{K}$ indeed is a Lipschitz constant for $L(\theta)$, hence the result now follows from Condition 4.1. $\square$

5 Conclusion

This paper treated functional variations in processes associated with Continuous Flow Models (CFMs). Specifically, it considered the effects of variations in defining processes (the inflow rate process, service rate process, and buffer size process) on various derived processes, including the workload processes and the overflow process. The paper obtained upper bounds on variations in the derived processes in terms of functional norms and Lipschitz constants. The analysis was first performed on single-flow CFMs and then extended to multi-flow CFMs with various service disciplines expressed as multiplexing priority rules. The variational bounds were then used to establish the unbiasedness of IPA derivatives for workload and loss-volume related functions.

Acknowledgments

We would like to thank the anonymous referees for numerous helpful comments.

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A Appendix

This appendix provides proofs to Proposition 3.2 and Proposition 3.3. We start with Proposition 3.2, whose proof relies on Lemma 3.2 and two additional preliminary results, Lemma A.1 and Lemma A.2.

Lemma A.1 Suppose that $\Delta\beta(t) \geq 0$ for all $t \in [0,T]$. Then, for every $\phi \in [0,T)$ and $\psi \in (\phi, T)$,

$$\Delta x(\psi) \; - \; \Delta x(\phi) \; \geq \; -\int_{\phi}^{\psi} \Delta\beta(\tau)\, d\tau. \tag{A.1}$$

Proof. Fix $\phi \in [0,T)$ and $\psi \in (\phi, T]$. By Assumption 3.1, there exists a finite partition $[\phi,\psi) = \cup_{j=1}^{k}[t_j, t_{j+1})$, such that for every $j = 1,\ldots,k$, exactly one of the following three conditions holds for all $t \in (t_j, t_{j+1})$:
(i) $x(t) + \Delta x(t) = c(t)$
(ii) $x(t) = 0$
(iii) $x(t) + \Delta x(t) < c(t)$ and $x(t) > 0$
In either case, we next show that

$$\Delta x(t_{j+1}) \; - \; \Delta x(t_j) \; \geq \; -\int_{t_j}^{t_{j+1}} \Delta\beta(\tau)\, d\tau, \quad j = 1,\ldots,k. \tag{A.2}$$

In Case *(i)*, we infer from part (a) of Lemma 3.2 that $x(t) = c(t)$, $t \in (t_j, t_{j+1})$. Consequently, $\Delta x(t_j) = \Delta x(t_{j+1}) = 0$. Since $\Delta\beta(\cdot) \geq 0$ by assumption, Eq. (A.2) immediately follows.
In Case *(ii)*, we infer from part (a) of Lemma 3.2 that $x(t) + \Delta x(t) = 0$, $t \in (t_j, t_{j+1})$. Consequently, $\Delta x(t_j) = \Delta x(t_{j+1}) = 0$, and since $\Delta\beta(\cdot) \geq 0$, Eq. (A.2) follows.
In Case *(iii)*, the application of part (a) of Lemma 2.2 to $x(\cdot) + \Delta x(\cdot)$, and the application of part (b) of Lemma 2.2 to $x(\cdot)$, yield the following two inequalities

$$\begin{aligned} x(t_{j+1}) + \Delta x(t_{j+1}) \; - \; (x(t_j) + \Delta x(t_j)) \; &\geq \; \int_{t_j}^{t_{j+1}} [\alpha(\tau) - (\beta(\tau) + \Delta\beta(\tau))]\, d\tau \\ x(t_{j+1}) \; - \; x(t_j) \; &\leq \; \int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)]\, d\tau \end{aligned}$$

Eq. (A.2) follows from the inequalities above by subtraction.

Finally, Eq. (A.1) follows by summing the inequality (A.2) over $j = 1,\ldots,k$, and noting that $\cup_{j=1}^{k}[t_j, t_{j+1}) = [\phi, \psi)$, $\square$

Lemma A.2 Suppose that $\Delta\beta(t) \geq 0$ for all $t \in [0,T]$. Let $(u,v) \subset [0,T]$ be an interval, such that $0 = x(t) + \Delta x(t) < x(t)$, for all $t \in (u,v)$. Let $[u,v) = \cup_{j=1}^{m}[t_j, t_{j+1})$ be a finite partition of the interval $[u,v)$. Let Φ and Ψ be two integer sets, such that $\Phi \subset \{1,\ldots,m\}$ and $\Psi = \{1,\ldots,m\} \setminus \Phi$. Then,

$$\sum_{j\in\Psi} \int_{t_j}^{t_{j+1}} [\beta(\tau) - \alpha(\tau)]\, d\tau \; \leq \; \sum_{j\in\Phi} \int_{t_j}^{t_{j+1}} \Delta\beta(\tau)\, d\tau \; + \; x(u) \; - \; x(v). \tag{A.3}$$

Proof. Since $x(t) > 0$ for all $t \in (u, v)$, an application of part (b) of Lemma 2.2 yields the inequality

$$x(t_{j+1}) \; - \; x(t_j) \; \le \; \int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)] \, d\tau, \quad j = 1, \ldots, m. \tag{A.4}$$

Subtracting and adding $\beta(\tau) + \Delta\beta(\tau)$ in the above integrand, we obtain,

$$x(t_{j+1}) \; - \; x(t_j) \; \le \; \int_{t_j}^{t_{j+1}} [\alpha(\tau) - (\beta(\tau) + \Delta\beta(\tau))] \, d\tau \; + \; \int_{t_j}^{t_{j+1}} \Delta\beta(\tau) \, d\tau.$$

Since by assumption, $x(t) + \Delta x(t) = 0$ for all $t \in [u, v]$, it follows from Eq. (2.1) that $\alpha(\tau) - (\beta(\tau) + \Delta\beta(\tau)) \le 0$, $\tau \in [t_j, t_{j+1}]$, and hence,

$$x(t_{j+1}) \; - \; x(t_j) \; \le \; \int_{t_j}^{t_{j+1}} \Delta\beta(\tau) \, d\tau, \quad j = 1, \ldots, m. \tag{A.5}$$

Next, sum Eq. (A.4) over $j \in \Psi$, and sum Eq. (A.5) over $j \in \Phi$, yielding

$$\sum_{j \in \Psi} (x(t_{j+1}) \; - \; x(t_j)) \; + \; \sum_{j \in \Phi} (x(t_{j+1}) - x(t_j)) \; \le$$
$$\sum_{j \in \Psi} \int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)] \, d\tau \; + \; \sum_{j \in \Phi} \int_{t_j}^{t_{j+1}} \Delta\beta(\tau) \, d\tau.$$

Recalling that $\Phi \cup \Psi = \{1, \ldots, m\}$, while $\Phi \cap \Psi = \emptyset$, we observe that the lefthand side of the above inequality is equal to $x(v) - x(u)$. Hence

$$x(v) \; - \; x(u) \; \le \; \sum_{j \in \Psi} \int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)] \, d\tau \; + \; \sum_{j \in \Phi} \int_{t_j}^{t_{j+1}} \Delta\beta(\tau) \, d\tau,$$

which is equivalent to (A.3). □

Proof of Proposition 3.2. Assume first that $\Delta\beta(t) \ge 0$ for all $t \in [0, T]$. We first prove the proposition under this assumption, and then prove the general case. Consider first part (a). Since $\Delta\beta(t) \ge 0$ for all $t \in [0, T]$, we have $\Delta x(t) \le 0$ for all $t \in [0, T]$ by part (a) of Lemma 3.2. From Lemma A.1 and the fact that $\Delta x(0) = 0$, it follows that for every $t \in [0, T]$,

$$0 \; \ge \; \Delta x(t) \; \ge \; -\int_0^t \Delta\beta(\tau) \, d\tau \; \ge \; -\int_0^T \Delta\beta(\tau) \, d\tau.$$

Therefore, $|\Delta x(t)| \le ||\Delta\beta||_1$, and maximizing over $t \in [0, T]$ yield part (a).

We next prove part (b) under the assumption that $\Delta\beta(t) \ge 0$ for all $t \in [0, T]$. The interval $[0, T]$ consists of alternating busy periods and idle periods of CFM_N. Let $[\phi, \psi]$ be any idle period of CFM_N, and let $[\eta, \rho]$ be any busy period of CFM_N. Clearly, the validity of part (b) is equivalent to the validity of the two inequalities

$$\int_\phi^\psi |\Delta\delta(\tau)| \, d\tau \;\; \le \;\; 2 \int_\phi^\psi \Delta\beta(\tau) \, d\tau, \tag{A.6}$$

$$\int_\eta^\rho |\Delta\delta(\tau)| \, d\tau \;\; \le \;\; 2 \int_\eta^\rho \Delta\beta(\tau) \, d\tau. \tag{A.7}$$

We begin with (A.6). By part (a) of Lemma 3.2, the interval $[\phi, \psi]$ is contained in an idle period of CFM_P. Using this fact in conjunction with Eq. (2.2), we deduce that $\delta(t) + \Delta\delta(t) = \delta(t) = \alpha(t)$ for all $t \in (\phi, \psi)$. This readily implies (A.6), because the lefthand side vanishes, while the righthand side is non-negative by the assumption $\Delta\beta(t) \geq 0$ for all $t \in [0, T]$.

We next prove (A.7). Since $[\eta, \rho]$ is a busy period of CFM_N, we have $x(t) > 0$ for all $t \in (\eta, \rho)$. Furthermore, the interval $[\eta, \rho]$ in CFM_P can be decomposed into alternating idle periods and busy periods. Without loss of generality, assume that the first and last periods of this decomposition are idle periods of CFM_P. That is, for some $k \geq 1$,

$$[\eta, \rho) \;=\; [u_1, v_1) \cup [v_1, u_2) \cup \cdots \cup [v_{k-1}, u_k) \cup [u_k, v_k), \tag{A.8}$$

where $[u_i, v_i]$, $i = 1, \ldots, k$, are idle periods of CFM_P, and $[v_i, u_{i+1}]$, $i = 1, \ldots, k-1$, are busy periods of CFM_P. Now, for every busy period $[v_i, u_{i+1}]$ of CFM_P, Eq. (2.2) implies that for all $t \in [v_i, u_{i+1}]$, $\delta(t) + \Delta\delta(t) = \beta(t) + \Delta\beta(t)$ and $\delta(t) = \beta(t)$. Combining this fact with the assumption $\Delta\beta(t) \geq 0$, for all $t \in [0, T]$, we obtain

$$\int_{v_i}^{u_{i+1}} |\Delta\delta(\tau)|\, d\tau \;=\; \int_{v_i}^{u_{i+1}} \Delta\beta(\tau)\, d\tau. \tag{A.9}$$

Next, consider an idle period $[u_i, v_i]$ of CFM_P for a given fixed $i \in \{1, \ldots, k\}$. By Assumption 3.1, there exists a finite partition $[u_i, v_i) = \cup_{j=1}^{m} [t_j, t_{j+1})$ of the interval $[u_i, v_i)$, such that for every $j = 1, \ldots, m$, exactly one of the following two conditions hold for every $t \in [t_j, t_{j+1})$:
(i) $\alpha(t) \geq \beta(t)$,
(ii) $\alpha(t) < \beta(t)$.
Define

$$\begin{aligned}
\Phi &= \{j \in \{1, \ldots, m\} :\; \alpha(t) \geq \beta(t),\; t \in [t_j, t_{j+1})\} \\
\Psi &= \{j \in \{1, \ldots, m\} :\; \alpha(t) < \beta(t),\; t \in [t_j, t_{j+1})\}
\end{aligned}$$

and observe that Lemma A.2 yields the inequality

$$\sum_{j \in \Psi} \int_{t_j}^{t_{j+1}} [\beta(\tau) - \alpha(\tau)]\, d\tau \;\leq\; \sum_{j \in \Phi} \int_{t_j}^{t_{j+1}} \Delta\beta(\tau)\, d\tau \;+\; x(u_i) \;-\; x(v_i). \tag{A.10}$$

Now for every $t \in (u_i, v_i)$, $0 = x(t) + \Delta x(t) < x(t)$ by assumption. It therefore follows from Eq. (2.2) that $\delta(t) + \Delta\delta(t) = \alpha(t)$ and $\delta(t) = \beta(t)$, and hence $\Delta\delta(t) = \alpha(t) - \beta(t)$.

Consider first $t \in (t_j, t_{j+1})$ for $j \in \Phi$. By definition of Φ, $\alpha(t) \geq \beta(t)$, whence we can write

$$|\Delta\delta(t)| \;=\; \alpha(t) - \beta(t).$$

Next, subtract and add $\beta(t) + \Delta\beta(t)$ to the equation above, and note that in view of Eq. (2.1), the assumption $x(t) + \Delta x(t) = 0$ implies $\alpha(t) \leq \beta(t) + \Delta\beta(t)$. It follows that $|\Delta\delta(t)| \leq \Delta\beta(t)$, whence integration yields

$$\int_{t_j}^{t_{j+1}} |\Delta\delta(\tau)|\, d\tau \;\leq\; \int_{t_j}^{t_{j+1}} \Delta\beta(\tau)\, d\tau, \quad j \in \Phi. \tag{A.11}$$

Consider next $t \in (t_j, t_{j+1})$ for $j \in \Psi$. As before, $\Delta\delta(t) = \alpha(t) - \beta(t)$, but now $\Delta\delta(t) < 0$ by definition of Ψ. Hence, $|\Delta\delta(t)| = \beta(t) - \alpha(t)$, which on substitution into the lefthand side of (A.10) yields

$$\sum_{j\in\Psi}\int_{t_j}^{t_{j+1}} |\Delta\delta(\tau)|d\tau \ \leq\ \sum_{j\in\Phi}\int_{t_j}^{t_{j+1}} \Delta\beta(\tau)\,d\tau \ +\ x(u_i)\ -\ x(v_i). \tag{A.12}$$

Since $[u_i, v_i) = \cup_{j=1}^{m}[t_j, t_{j+1})$ is a partition, Eqs. (A.11) and (A.12) now imply that

$$\int_{u_i}^{v_i} |\Delta\delta(\tau)|\,d\tau \ \leq\ 2\sum_{j\in\Phi}\int_{t_j}^{t_{j+1}} \Delta\beta(\tau)\,d\tau \ +\ x(u_i)\ -\ x(v_i),$$

and since $\cup_{j\in\Phi}[t_j, t_{j+1}) \subset [u_i, v_i)$, we consequently have the inequality

$$\int_{u_i}^{v_i} |\Delta\delta(\tau)|d\tau \ \leq\ 2\int_{u_i}^{v_i} \Delta\beta(\tau)\,d\tau \ +\ x(u_i)\ -\ x(v_i). \tag{A.13}$$

Combining Eqs. (A.8), (A.9) and (A.13), it follows that

$$\int_{\eta}^{\rho} |\Delta\delta(\tau)|\,d\tau \ \leq\ \sum_{i=1}^{k-1}\int_{v_i}^{u_{i+1}} \Delta\beta(\tau)\,d\tau + 2\sum_{i=1}^{k}\int_{u_i}^{v_i} \Delta\beta(\tau)\,d\tau + \sum_{i=1}^{k}(x(u_i) - x(v_i)), \tag{A.14}$$

where $u_1 = \eta$ and $v_k = \rho$. Since the interval $[\eta, \rho]$ constitutes a busy period for CFM_N, one has $x(\eta) = 0$, while $x(\rho) \geq 0$ (the case $x(\rho) > 0$ can arise when $\rho = T$). Therefore, the last sum in (A.14) can be rearranged to yield the following inequality:

$$\begin{aligned}\sum_{i=1}^{k}(x(u_i) - x(v_i)) \ &=\ x(\eta)\ +\ \sum_{i=2}^{k}\left[x(u_i) - x(v_{i-1})\right]\ -\ x(\rho) \\ &\leq\ \sum_{i=2}^{k}\left[x(u_i) - x(v_i)\right].\end{aligned} \tag{A.15}$$

The interval $[v_{i-1}, u_i]$, $i = 2, \ldots, k$, constitutes a busy period of CFM_P, which starts at v_{i-1} and ends at $u_i < \rho$, so that $x(v_{i-1}) + \Delta x(v_{i-1}) = x(u_i) + \Delta x(u_i) = 0$. Hence, $\Delta x(v_{i-1}) = -x(v_{i-1})$ and $\Delta x(u_i) = -x(u_i)$. An application of Lemma A.1 now yields, for all $i = 2, \ldots, k$,

$$x(u_i)\ -\ x(v_{i-1})\ =\ -[\Delta x(u_i) - \Delta x(v_{i-1})]\ \leq\ \int_{v_{i-1}}^{u_i} \Delta\beta(\tau)\,d\tau. \tag{A.16}$$

Eqs. (A.14), (A.15) and (A.16) now lead to the inequality

$$\int_{\eta}^{\rho} |\Delta\delta(\tau)|\,d\tau \ \leq\ 2\sum_{i=1}^{k}\int_{u_i}^{v_i} \Delta\beta(\tau)\,d\tau \ +\ 2\sum_{i=2}^{k}\int_{v_{i-1}}^{u_i} \Delta\beta(\tau)\,d\tau,$$

which in view of Eq. (A.8) implies Eq. (A.7). This completes the proof of part (b) under the assumption that $\Delta\beta(t) \geq 0$ for all $t \in [0, T]$.

We now prove part (c) under the assumption that $\Delta\beta(t) \geq 0$ for all $t \in [0, T]$. Applying Lemma 2.1 separately to CFM_P and CFM_N, and taking differences yields the following equation,

$$\int_0^T \Delta\delta(\tau)\, d\tau \;+\; \int_0^T \Delta\gamma(\tau)\, d\tau \;+\; \Delta x(T) \;=\; 0.$$

Part (c) now follows directly from Lemma 3.2 and the inequalities in part (a) and part (b).

We finally prove the proposition for the general case, where $\Delta\beta(t)$ may be negative. Similarly to the arguments in the proof of Proposition 3.1, define

$$\begin{aligned} \beta_{min}(t) &= \min\{\beta(t), \beta(t) + \Delta\beta(t)\} \\ \beta_{max}(t) &= \max\{\beta(t), \beta(t) + \Delta\beta(t)\}. \end{aligned}$$

Moreover, for any derived process $z(t)$, let $z_{min}(t)$ and $z_{max}(t)$ denote the process $z(t)$ resulting from $\beta_{min}(t)$ and $\beta_{max}(t)$, respectively.

Now, in view of Lemma 3.2, the proofs of parts (a) and (c) of the present proposition are analogous to the proof of Proposition 3.1 from the inequalities in (3.2) and (3.4), respectively. Consequently, these proofs are omitted.

It remains to prove part (b) for the general case. The triangle inequality yields

$$||\Delta\delta||_1 \;\leq\; ||\delta \;+\; \Delta\delta - \delta_{max}||_1 \;+\; ||\delta_{max} - \delta||_1. \tag{A.17}$$

Since $\beta_{max}(t) \geq \beta(t) + \Delta\beta(t)$ and $\beta_{max}(t) \geq \beta(t)$, $t \in [0, T]$, an application of this proposition under the monotonicity conditions earlier proved, yields the inequalities

$$\begin{aligned} ||\delta_{max} - (\delta + \Delta\delta)||_1 &\leq 2||\beta_{max} - (\beta + \Delta\beta)||_1 \\ ||\delta_{max} - \delta||_1 &\leq 2||\beta_{max} - \beta||_1. \end{aligned}$$

Utilizing these inequalities in Eq. (A.17) yields

$$||\Delta\delta||_1 \;\leq\; 2\left(||\beta_{\max} - (\beta + \Delta\beta)||_1 \;+\; ||\beta_{\max} - \beta||_1\right). \tag{A.18}$$

But by definition of β_{max}, we can write

$$|\beta_{\max}(t) - (\beta(t) + \Delta\beta(t)| \;+\; |\beta_{\max}(t) - \beta(t)| \;=\; |\Delta\beta(t)|, \quad t \in [0, T].$$

This, together with Eq. (A.18), implies the inequality

$$||\Delta\delta||_1 \;\leq\; 2||\Delta\beta||_1,$$

finally yielding part (b) and hence completing the proof. $\square$.

We next turn to proving Proposition 3.3. The following lemma will be needed.

Lemma A.3 Suppose Assumption 3.1 is in force. If $\Delta c(t) \geq 0$ for every $t \in [0,T]$, then
(a) $\Delta x(t) \geq 0$
(b) $\Delta\delta(t) \geq 0$
(c) $\int_0^t \Delta\gamma(\tau)\, d\tau \leq 0$, for all $t \in [0,T]$.

Proof. Part (a) and (b) are direct consequences of Eqs. (2.1) and (2.2).

To prove part (c), apply Lemma 2.1 to CFM_N and CFM_P, and take differences, yielding

$$\int_0^t \Delta\gamma(\tau)\, d\tau \;+\; \int_0^t \Delta\delta(\tau)\, d\tau \;+\; \Delta x(t) \;=\; 0, \quad t \in [0,T].$$

Part (c) now follows, since the last two terms above are non-negative by part (a) and part (b). □

Proof of Proposition 3.3. By the monotonicity properties of Lemma A.3, it suffices to consider the case where $\Delta c(t) \geq 0$ for all $t \in [0,T]$, and prove the following three inequalities,

$$\Delta x(T) \;\leq\; ||\Delta c||_\infty \tag{A.19}$$

$$\int_0^T \Delta\gamma(\tau)\, d\tau \;\geq\; -K\, ||\Delta c||_\infty \tag{A.20}$$

$$\int_0^T \Delta\delta(\tau)\, d\tau \;\leq\; (K+1)\, ||\Delta c||_\infty \tag{A.21}$$

Consider first the inequality (A.19). By Assumption 3.1, there exists a finite partition $[0,T) = \cup_{j=1}^m [t_j, t_{j+1})$ of the interval $[0,T)$, such that for every $j = 1, \ldots, m$, exactly one of the following four cases holds for every $t \in (t_j, t_{j+1})$:
(i) $x(t) = c(t)$
(ii) $x(t) < c(t)$ and $x(t) + \Delta x(t) = c(t) + \Delta c(t)$
(iii) $0 < x(t) < c(t)$ and $x(t) + \Delta x(t) < c(t) + \Delta c(t)$
(iv) $x(t) = 0$ and $x(t) + \Delta x(t) < c(t) + \Delta c(t)$
For Case (i), we show that $\Delta x(t_{j+1}) \leq \Delta c(t_{j+1})$, while for Cases (ii), (iii) and (iv), we show that $\Delta x(t_{j+1}) \leq \Delta x(t_j)$. Since by assumption $\Delta x(0) = 0$, $t_1 = 0$ and $t_{m+1} = T$, Eq. (A.19) will then follow.

In Case (i), we combine the general inequality $x(t) + \Delta x(t) \leq c(t) + \Delta c(t)$ with the assumption $x(t) = c(t)$, and deduce by subtraction that $\Delta x(t_{j+1}) \leq \Delta c(t_{j+1})$.

In Case (ii), we apply part (a) of Lemma 2.2 to $x(\cdot)$, yielding

$$x(t_{j+1}) \;-\; x(t_j) \;\geq\; \int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)]\, d\tau.$$

By assumption, the buffer is full in CFM_P, throughout the interval (t_j, t_{j+1}) . Thus, by Eq. (2.1), we necessarily have $\alpha(\tau) - \beta(\tau) \;\geq\; \dot{c}(\tau) + \Delta\dot{c}(\tau)$ for all $\tau \in (t_j, t_{j+1})$, and substituting this inequality into the previous one yields

$$x(t_{j+1}) \;-\; x(t_j) \;\geq\; \int_{t_j}^{t_{j+1}} [\dot{c}(\tau) + \Delta\dot{c}(\tau)]\, d\tau \;=\; c(t_{j+1}) + \Delta c(t_{j+1}) - (c(t_j) + \Delta c(t_j)).$$

But $c(t_{j+1})+\Delta c(t_{j+1}) = x(t_{j+1})+\Delta x(t_{j+1})$ and $c(t_j)+\Delta c(t_j) = x(t_j)+\Delta x(t_j)$ imply

$$x(t_{j+1}) - x(t_j) \;\geq\; x(t_{j+1}) + \Delta x(t_{j+1}) - (x(t_j) + \Delta x(t_j)),$$

namely, $\Delta x(t_{j+1}) \leq \Delta x(t_j)$.

In Case (iii) the buffer is neither full nor empty throughout the interval (t_j, t_{j+1}) in either CFM_N or CFM_P (the assumption that $x(t) > 0$ implies $x(t)+\Delta x(t) > 0$ by part (a) of Lemma A.3). Therefore, Eq. (2.1) implies that the workload increments in CFM_N and CFM_P are identical over (t_j, t_{j+1}), namely,

$$x(\tau) - x(t_j) \;=\; x(\tau) + \Delta x(\tau) - (x(t_j) + \Delta x(t_j)), \quad \tau \in (t_j, t_{j+1}).$$

The equation above implies $\Delta x(t_{j+1}) = \Delta x(t_j)$.

Finally, consider Case (iv), where the buffer is empty in CFM_N and not full in CFM_P. Regarding CFM_N, $x(t_j) = x(t_{j+1}) = 0$, and by Eq. (2.1), $\alpha(t) - \beta(t) \leq 0$ for all $t \in [t_j, t_{j+1}]$. Therefore, Eq. (2.1) implies to CFM_P that $x(t_{j+1}) + \Delta x(t_{j+1}) \leq x(t_j) + \Delta x(t_j)$. Having seen that $x(t_{j+1}) = x(t_j) = 0$, it now follows that $\Delta x(t_{j+1}) \leq \Delta x(t_j)$. This completes the proof of the inequality $\Delta x(t_{j+1}) \leq \Delta x(t_j)$ in Cases (ii), (iii) and (iv), and hence the proof of Eq. (A.19).

Next, consider Eq. (A.20). Let the interval $[t_B, t_E]$ be a busy period in CFM_N. By Assumption 3.1 there exists a finite partition $[t_B, t_E) = \cup_{j=1}^{m} [t_j, t_{j+1})$ of the interval $[t_B, t_E)$, such that for every $j = 1, \ldots, m$, exactly one of the following four conditions holds for all $t \in (t_j, t_{j+1})$:

(i) $x(t) = c(t)$ and $x(t) + \Delta x(t) = c(t) + \Delta c(t)$
(ii) $x(t) = c(t)$ and $x(t) + \Delta x(t) < c(t) + \Delta c(t)$
(iii) $x(t) < c(t)$ and $x(t) + \Delta x(t) = c(t) + \Delta c(t)$
(iv) $x(t) < c(t)$ and $x(t) + \Delta x(t) < c(t) + \Delta c(t)$

We now proceed to prove that each of the four cases above satisfies the equation

$$\int_{t_j}^{t_{j+1}} \Delta\gamma(\tau)\, d\tau \;=\; -(\Delta x(t_{j+1}) \;-\; \Delta x(t_j)). \tag{A.22}$$

In Case (i), Eq. (2.3) implies

$$\int_{t_j}^{t_{j+1}} \Delta\gamma(\tau)\, d\tau \;=\; -\int_{t_j}^{t_{j+1}} \Delta\dot{c}(\tau)\, d\tau \;=\; -(\Delta c(t_{j+1}) \;-\; \Delta c(t_j)),$$

and this result coincides with Eq. (A.22) by the conditions of Case (i).

In Case (ii), Eq. (2.3) implies

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \Delta\gamma(\tau)\, d\tau \;&=\; -\int_{t_j}^{t_{j+1}} \gamma(\tau)\, d\tau \;=\; -\int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau) - \dot{c}(\tau)]\, d\tau \\ &=\; -\int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)]\, d\tau \;+\; c(t_{j+1}) \;-\; c(t_j). \end{aligned} \tag{A.23}$$

Now, the interval (t_j, t_{j+1}) is contained in a busy period $[t_B, t_E]$ of CFM_N, and by part (a) of Lemma A.3 it is also contained in a busy period of CFM_P. Consequently, $x(t) + \Delta x(t) > 0$ for all $t \in (t_j, t_{j+1})$. Thus, the buffer is neither empty nor full in CFM_P (the latter by the condition defining Case (ii)). Therefore, Eq. (2.1) implies

$$\int_{t_j}^{t_{j+1}} [\alpha(\tau) - \beta(\tau)]\, d\tau \;=\; x(t_{j+1}) + \Delta x(t_{j+1}) \;-\; (x(t_j) + \Delta x(t_j)).$$

Eq. (A.22) now follows by substituting the equation above into Eq. (A.23), and noting the equalities $c(t_j) = x(t_j)$ and $c(t_{j+1}) = x(t_{j+1})$.

Case (iii) is analogous to Case (ii) and the same arguments apply in proving Eq. (A.22).

Finally, in Case (iv), the buffer never becomes full in the interval $t \in (t_j, t_{j+1})$ in CFM_N as well as in CFM_P. Since in both CFMs, the buffer also does not become empty (because $[t_j, t_{j+1}]$ is a subinterval of a busy period of CFM_N), applications of Eq. (2.1) to the two CFMs yields $\Delta x(t_{j+1}) - \Delta x(t_j) = 0$. But the lack of spillover implies $\int_{t_j}^{t_{j+1}} \Delta\gamma(\tau)\, d\tau = 0$, so that Eq. (A.22) clearly holds.

Having shown that Eq. (A.22) holds for each $j = 1, \ldots, m$, we next sum that equation over the range $j = 1, \ldots, m$, obtaining

$$\int_{t_B}^{t_E} \Delta\gamma(\tau)\, d\tau \;=\; -(\Delta x(t_E) - \Delta x(t_B)).$$

Since t_B starts a busy period in CFM_N, $x(t_B) = 0$, and hence $\Delta x(t_B) \geq 0$. This inequality and the latter equation clearly imply $\int_{t_B}^{t_E} \Delta\gamma(\tau)\, d\tau \geq -\Delta x(t_E)$. Furthermore, since $-\Delta x(t_E) \geq -||\Delta x||_\infty$, and $||\Delta x||_\infty \;\leq\; ||\Delta c||_\infty$ by (A.19), we have

$$\int_{t_B}^{t_E} \Delta\gamma(\tau)\, d\tau \;\geq\; -||\Delta c||_\infty. \tag{A.24}$$

Recall that $K = \max\{K_N, K_P\}$. Then, clearly Eq. (A.24) yields Eq. (A.20).

Finally, to prove Eq. (A.21), observe that Lemma 2.1 implies

$$\int_0^T \Delta\delta(\tau)\, d\tau \;+\; \int_0^T \Delta\gamma(\tau)\, d\tau \;+\; \Delta x(T) \;=\; 0.$$

Eq. (A.21) now readily follows with the aid of Eqs. (A.19) and (A.20). This completes the proof of the Proposition. □