THE RUN PROBABILITIES OF TES PROCESSES

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ABSTRACT

The run statistics of a discrete-time, real-valued stochastic process are the statistics of process excursions above a given level. As such they are a special case of first passage times (hitting times) in discrete time. The study of run probabilities is motivated by applications such as compressed video, where a random and autocorrelated sequence of compressed frames arrives deterministically at a finite buffer, and the loss probability of consecutive frames (runs) constitutes a better measure of service quality than simple loss probabilities. This paper studies the run probabilities of a subclass of TES processes. A uniform TES process is a modulo-1 autoregressive stochastic process, uniform on $[0, 1)$; general TES processes are obtained by transforming a basic TES process to ones with general marginals. The paper develops an integral equation in the generating function of the run probabilities of TES processes. An exact matrix solution of theoretical interest is obtained, but the solution requires expensive computations including inverting and margining. We, therefore, derive additionally a Sokolov-type method to obtain numerical approximations. We also identify a class of TES processes for which a closed-form solution can be exhibited. Finally, we give some numerical examples that support the efficacy of our approach by comparing the approximate run probabilities to Monte Carlo simulation statistics, as well as the exact solution in the transform domain.

Keywords and Phrases: TES Processes, basic TES processes, run probabilities, stochastic processes, Sokolov approximation methodology.

1 INTRODUCTION

Let $\{X_n\}_{n=0}^{\infty}$ be a discrete-time, real-valued stationary stochastic process. The *run probabilities* of $\{X_n\}$ above level b (b real) and length r ($r \geq 0$) are defined by

$$H_X(r,b) \;=\; \begin{cases} P\{X_n \leq b\}, & r = 0 \\ P\{X_n > b, \ldots, X_{n+r-1} > b, X_{n+r} \leq b\}, & r > 0 \end{cases} \tag{1.1}$$

while the corresponding run probabilities of length greater than r are defined by

$$\bar{H}_X(r,b) \;=\; P\{X_n > b, X_{n+1} > b, \ldots, X_{n+r} > b\}. \tag{1.2}$$

By stationarity of $\{X_n\}$, the righthand sides of Eqs. (1.1) and (1.2) are invariant under translations of n. Clearly, for every real b,

$$\sum_{n=0}^{\infty} H_X(n,b) \;=\; 1 \;-\; \bar{H}_X(\infty,b). \tag{1.3}$$

We mention that the dual case of runs below a given level has an analogous mathematical structure to runs above a given level; consequently, we restrict the treatment to the latter.

The concept of runs is a special case of first passage times (or hitting times) in discrete time; see, e.g., Aldous (1989), Erdös and Rényi (1976), Keilson (1979) and Leadbetter (1983), which treat excursions and their extremal statistics in processes with renewal and Markovian structure. The zero-run probability, $H_X(0,b)$, corresponds to the event that an observation of the process $\{X_n\}$ finds it at or below level b, whereas the positive-run probability, $H_X(r,b),\ r > 0$, corresponds to the event that r successive observations of $\{X_n\}$ will lie above level b and the next one will lie at or below it. The r successive observations then form a run of length r above level b. The run notion involves only future observations relative to a starting time n, without regard to observations made prior to n.

To motivate the calculation of run probabilities, suppose that $\{X_n\}$ models the amount of work arriving at a time-slotted queue with buffer capacity b. Suppose, further, that work not processed by the server at each time slot is cleared (lost). Then $H_X(r,b)$ models the probability that r successive work arrivals will suffer a loss. A current emerging application for this model is transmission of coded (compressed) video. Here work arrives deterministically at the transmission buffer in the form of coded video frames at the rate of some 30 frames per second, but the amount of work is random. Suppose that transmission is modeled as a server whose work rate (per 1/30 second interval) is b. The server can then successfully transmit a coded frame whose size is at most b, and coded frames exceeding this size are considered damaged or lost. In this context, the run probabilities, $H_X(r, b)$, provide a measure of perceived video quality at the destination end. Furthermore, measuring this quality of service in terms of runs of damaged frames is better than the standard measure in terms of the fraction of frames damaged or lost. Notice that the former measure uses detailed information on the joint probabilities of $\{X_n\}$, whereas the latter only uses its marginal statistics.

If the process $\{X_n\}$ were a renewal process, then the quantities, $H_X(r, b)$, could be easily computed from the marginal statistics as geometric probabilities. However, the process $\{X_n\}$ is generally autocorrelated, thereby rendering the calculation of the $H_X(r, b)$ a non-trivial task. In practice, traffic in high-speed telecommunications networks (such as file transfers and compressed video) can be rather bursty due to the presence of significant positive autocorrelations in interarrival intervals; see, e.g., Lee et al. (1992) and Melamed et al. (1992). As a rule, the effect of positive autocorrelations in arrivals and services is to severely degrade queueing performance measures, as compared to classical queueing with renewal (GI) counterparts; see, e.g., Livny et al. (1993) and references cited therein. We are, therefore, strongly motivated to study traffic and workload models which can capture temporal dependencies, and in particular, a wide range of autocorrelation structures commonly encountered in practice. Moreover, the need to simultaneously capture the marginal distribution and autocorrelation function of stochastic traffic processes will further motivate us here to model the underlying process, $\{X_n\}$, as a TES (*Transform-Expand-Sample*) process, to be described in the next section; for more details, refer to Melamed (1991), Jagerman and Melamed (1992a) and (1992b).

The rest of the paper is organized as follows. Section 2 contains some preliminaries. Section 3 formulates the run-probabilities problem as an integral equation in a generating function. An exact matrix solution of theoretical interest is derived in Section 4, and an approximation utilizing the exponential-subspace Sokolov method, is obtained in Section 5 as a practical method for computing run probabilities. Section 6 identifies a class of TES processes for which a closed-form solution can be exhibited. Finally, Section 7 contains examples which support the efficacy of our approach and the conclusions of this paper.

2 PRELIMINARIES

This section summarizes briefly some preliminary material. It puts together relevant information on TES processes, Fredholm equations and Sokolov approximations.

2.1 TES Processes

A detailed study of TES processes may be found in Melamed (1991), Jagerman and Melamed (1992a), (1992b) and (1994). Here we consider basic TES processes $\{U_n^+\}_{n=0}^{\infty}$, defined by the autoregressive scheme with modulo-1 reduction

$$U_n^+ = \begin{cases} U_0, & n = 0 \\ \langle U_{n-1}^+ + V_n \rangle, & n > 0 \end{cases} \tag{2.1}$$

The plus subscript is used to distinguish this process from other flavors of TES processes; see Jagerman and Melamed (1992a). In Eq. (2.1), the initial variate U_0 is distributed uniformly on the interval $[0, 1)$ (denoted by $U_0 \sim$ Uniform$[0, 1)$), and $\{V_n\}_{n=1}^{\infty}$ is an arbitrary i.i.d. (*independent identically distributed*) sequence with density function f_V; the V_n form a sequence of *innovations*, i.e., for each $n \geq 1$, V_n is independent of $\{U_0^+, U_1^+, \ldots, U_{n-1}^+\}$. The angular brackets denote the fractional part operator, defined for any real x by $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor =$ max$\{n$ integer : $n \leq x\}$ is the integral part of x. Note, that $\langle x \rangle \geq 0$, even for negative x.

It was shown, in Jagerman and Melamed (1992a), that $\{U_n^+\}_{n=0}^{\infty}$ is a stationary Markovian ergodic sequence with the Uniform$[0, 1)$ marginal distribution. Furthermore, TES processes of the form (2.1) have a simple geometric interpretation (*ibid.*) as random walks on a circle with circumference 1, and step size density f_V. The τ-step transition density, $g_\tau^+(y|x)$ of $\{U_n^+\}$, has the representation (*ibid.*),

$$g_\tau^+(y|x) = \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^{\tau}(i2\pi\nu)\, e^{i2\pi\nu(y-x)}, \quad 0 \leq x, y < 1, \tag{2.2}$$

where $\tilde{f}_V$ is the bilateral Laplace transform $\tilde{f}_V(s) = \int_{-\infty}^{\infty} e^{-sx} f_V(x)\,dx$ of f_V, and $\tilde{f}_V^{\tau}$ is its τ-th power, corresponding to the τ-fold convolution of f_V. We observe that the auxiliary function,

$$g(x) = \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, e^{i2\pi\nu x}, \tag{2.3}$$

is related to the one-step transition density by

$$g_1^+(y|x) = g(y-x). \tag{2.4}$$

Since Eq. (2.3) is the Fourier representation of $g(x)$, it follows from the Poisson summation formula (Lighthill (1960)) that $g(x)$ can be represented in terms of f_V as

$$g(x) = \sum_{\nu=-\infty}^{\infty} f_V(x+\nu), \tag{2.5}$$

because $g(x)$ is periodic with period one. From Eq. (2.5) it follows that $g(x)$ is in fact the common density function of the $\langle V_n \rangle$.

In practice, one is actually interested in transforming the uniform TES process $\{U_n^+\}_{n=0}^{\infty}$ into a new process $\{X_n^+\}_{n=0}^{\infty}$ via

$$X_n^+ = D(U_n^+), \quad n \geq 0, \tag{2.6}$$

where D is a real measurable function on $[0, 1)$, called a *distortion*. The sequences $\{U_n^+\}$ and $\{X_n^+\}$ are called, respectively, the *background* and *foreground* TES sequences, though we sometimes omit the qualification and call each of them a TES sequence whenever the context is clear. In applications, it is common to take $D = F^{-1}$, where F is some marginal distribution, obtained in practice from sample data, usually in the form of an empirical histogram. More precisely, we define $F^{-1}(y) = \inf\{x : F(x) = y\}$, and this definition is valid even if F is not one-one (F is always monotone increasing, but not necessarily strictly monotone). For a distortion of the form $D = F^{-1}$, the process $\{X_n^+\}$ will have marginal distribution F, and in particular, this allows us to model and match any empirical distribution. More complicated distortions are discussed in Jagerman and Melamed (1992a) and (1992b).

2.2 Fredholm Integral Equations and the Enskog Relation

Consider the Fredholm integral equation of second kind (Tricomi (1957), Chapter 3)

$$\phi(y) \;=\; f(y) + \lambda \int_c^d \phi(x)\, K(x,y)\, dx, \quad c \leq y \leq d, \tag{2.7}$$

where $\phi(y)$ is the unknown function, $f(y)$ is the forcing function, λ is a constant and $K(x,y)$ is a square-integrable kernel on the square $[c,d] \times [c,d]$. The Fredholm operator $\mathcal{F}$ maps a real square-integrable function $\xi(x)$ to a real function $\mathcal{F}[\xi]$, where

$$\mathcal{F}[\xi](y) \;=\; \xi(y) - \lambda \int_c^d \xi(x)\, K(x,y)\, dx. \tag{2.8}$$

The Fredholm integral equation (2.7) can then be written as

$$\mathcal{F}[\phi](y) \;=\; f(y). \tag{2.9}$$

Further, let $\mathcal{F}^*$ denote the associated (dual) Fredholm operator, defined by

$$\mathcal{F}^*[\xi](y) \;=\; \xi(y) - \lambda \int_c^d \xi(x)\, K(y,x)\, dx. \tag{2.10}$$

Note that Eq. (2.8) and (2.10) are similar, except that the arguments of the kernel are transposed. The Enskog relation (generalized Green's formula) relates the operators $\mathcal{F}$ and $\mathcal{F}^*$ through the identities

$$\int_c^d \mathcal{F}^*[\xi](y)\, \phi(y)\, dy \;=\; \int_c^d \mathcal{F}[\phi](y)\, \xi(y)\, dy \;=\; \int_c^d \xi(y)\, f(y)\, dy, \tag{2.11}$$

where the choice of $\xi(x)$ is arbitrary.

2.3 The General Sokolov Approximation Methodology

The *Sokolov approximation methodology* (see Luchka (1965), Chapter 1) constructs a sequence of functions, $\{\phi_n(y)\}_{n=0}^{\infty}$, that approximate the unknown function, $\phi(y)$, in the Fredholm integral equation (2.7) with the aid of a sequence of auxiliary functions, $\{\alpha_n(y)\}_{n=1}^{\infty}$. The approximation scheme is started off by selecting a vector space M of function elements, and some initial approximation $\phi_0(y)$. Thereafter, at each iteration step $n > 0$, we solve simultaneously for (ϕ_n, α_n) in the recursive equation system,

$$\begin{aligned}
\phi_n(y) &= f(y) + \lambda \int_c^d [\phi_{n-1}(x) + \alpha_n(x)]\, K(x,y)dx, && (2.12)\\
\alpha_n(y) &= P_M[\phi_n - \phi_{n-1}](y), && (2.13)
\end{aligned}$$

where P_M is a projection operator that maps a square-integrable real function to an element of M. Note that in Eq. (2.13), the function $P_M[\phi_n - \phi_{n-1}] \in M$ is obtained by projecting the difference function $\psi_n(y) = \phi_n(y) - \phi_{n-1}(y)$ on M. The Sokolov approximation methodology requires us to supply a choice of the initial approximation, $\phi_0(y)$, and the function vector space, M. Each such pair of choices determines the projection operator, P_M, the auxiliary function sequence, $\{\alpha_n(y)\}_{n=1}^{\infty}$, and the approximation sequence, $\{\phi_n(y)\}_{n=0}^{\infty}$. The choice of M and P_M is critical to the speed of convergence, though in practice, one often chooses P_M as the orthogonal projection. For example, if $M = \{\phi\}$ is the singleton space consisting of the zero element ϕ (the identically zero function), then $\alpha_n = \phi$ for all n, and the Sokolov recursive scheme (2.12)-(2.13) reduces to

$$\phi_n(y) = f(y) + \lambda \int_c^d \phi_{n-1}(x)\, K(x,y)\, dx,$$

which is just the Picard iteration scheme. On the other hand, if M can be chosen such that $\phi_n - \phi_{n-1} \in M$ for some n, then $\alpha_n = P_M[\phi_n - \phi_{n-1}] = \phi_n - \phi_{n-1}$, whence

$$\phi_n(y) = f(y) + \lambda \int_c^d \phi_n(x)\, K(x,y)\, dx,$$

so that $\phi = \phi_n$ is the solution of (2.7). Ideally, we would like that to happen for $n = 1$, implying that convergence to the solution is achieved in a single step!

2.4 The Exponential-Subspace Sokolov Approximation

In Section 6 it will be shown how to compute the run probabilities, $H_X(r, b)$, for TES processes with exponential innovation densities. This motivates the choice, in this section, of a function subspace consisting of exponential functions, henceforth to be referred to as an *exponential subspace*. To this end, we shall need an approximation to the dominant eigenvalue. Since we are only interested in probabilistic kernels, we shall henceforth assume that $K(x, y) \geq 0$. Consequently, the lowest eigenvalue,

λ_0, is positive and the corresponding eigenfunction, $\theta_0(y)$, satisfying

$$\theta_0(y) \;=\; \lambda_0 \int_c^d \theta_0(x)\, K(x,y)\, dx, \tag{2.14}$$

can be chosen to satisfy $\theta_0(y) \geq 0$. Let further

$$\bar{\theta}_0 \;=\; \frac{1}{d-c} \int_c^d \theta_0(y)\, dy \;=\; \lambda_0 \frac{1}{d-c} \int_c^d \int_c^d \theta_0(x)\, K(x,y)\, dx\, dy \tag{2.15}$$

be the mean value of $\theta_0(y)$. To approximate λ_0 by some $\hat{\lambda}_0$, it is reasonable to replace the detailed information provided by $\theta_0(x)$ by its mean, $\bar{\theta}_0$. Since $\bar{\theta}_0 > 0$, one gets

$$\lambda_0^{-1} \;\simeq\; \frac{1}{d-c} \int_c^d \int_c^d K(x,y)\, dx\, dy \;=\; \hat{\lambda}_0^{-1}. \tag{2.16}$$

Next, define M_E to be the subspace spanned by the exponential function,

$$\theta(y) \;=\; e^{\omega y}, \quad \omega \;=\; \ln(\hat{\lambda}_0). \tag{2.17}$$

The projection of an element $u(y)$ is, therefore,

$$P_M[u](y) \;=\; \beta\, \theta(y), \quad \beta \;=\; \frac{\displaystyle\int_c^d e^{\omega y}\, u(y)\, dy}{\displaystyle\int_c^d e^{2\omega y}\, dy} \;=\; \frac{2\omega}{e^{2\omega d} - e^{2\omega c}} \int_c^d e^{\omega y}\, u(y)\, dy. \tag{2.18}$$

Thus, taking the initial function, $\phi_0(y)$, to be the forcing function, $f(y)$, one has

$$\phi_1(y) - \phi_0(y) \;=\; \lambda \int_c^d [f(x) + \alpha_1(x)]\, K(x,y)\, dx, \tag{2.19}$$

$$\alpha_1(y) \;=\; \gamma\, \theta(y), \tag{2.20}$$

where from (2.13),

$$\begin{aligned} \gamma \;&=\; \frac{2\omega\lambda}{e^{2\omega d} - e^{2\omega c}} \int_c^d \int_c^d e^{\omega y}\, K(x,y)\, [f(x) + \gamma\, e^{\omega x}]\, dx\, dy \\ &=\; \frac{\displaystyle \frac{2\omega\lambda}{e^{2\omega d} - e^{2\omega c}} \int_c^d \int_c^d e^{\omega y}\, K(x,y)\, f(x)\, dx\, dy}{\displaystyle 1 - \frac{2\omega\lambda}{e^{2\omega d} - e^{2\omega c}} \int_c^d \int_c^d e^{\omega(x+y)}\, K(x,y)\, dx\, dy}. \end{aligned} \tag{2.21}$$

The approximation, $\phi_1(y)$, now follows immediately from Eqs. (2.12) and (2.19), namely,

$$\phi_1(y) \;=\; f(y) \;+\; \lambda \int_c^d f(x)\, K(x,y)\, dx \;+\; \lambda\, \gamma \int_c^d e^{\omega x}\, K(x,y)\, dx. \tag{2.22}$$

3 AN INTEGRAL EQUATION FORMULATION

In this section we derive an integral equation for the generating function of the run probabilities, $H_U^+(r,a)$, for the (uniform) background sequence $\{U^+\}$. Their foreground counterparts, $H_X^+(r,b)$, can be readily derived from the relation

$$H_X^+(r,b) \;=\; H_U^+(r,a), \tag{3.1}$$

where $b = F^{-1}(a)$. Since foreground sequences will, henceforth, be discussed infrequently, we shall routinely omit the U^+ subscripts in order to economize on notation, restoring them as necessary only to resolve ambiguities.

Accordingly, let $h(r,a,y)$ denote the function

$$h(r,a,y) \;=\; \begin{cases} 1, & r=0 \\ \dfrac{\partial}{\partial y}\, P\{U_0^+ > a, \ldots, U_{r-1}^+ > a, U_r \le y\}, & r>0,\;\; 0<y<1\,. \end{cases} \tag{3.2}$$

From Eq. (1.1), the run probabilities can be written as

$$H(r,a) \;=\; \int_0^a h(r,a,y)\,dy \;=\; P\{U_0^+ > a, \ldots, U_{r-1}^+ > a, U_r \le a\}, \tag{3.3}$$

and

$$\bar{H}(r,a) \;=\; \int_a^1 h(r,a,y)\,dy \;=\; P\{U_0^+ > a, \ldots, U_{r-1}^+ > a, U_r > a\}. \tag{3.4}$$

Next, we use the Markov property of $\{U_n^+\}$ and Eq. (2.4) to write the recursive scheme,

$$h(r,a,y) \;=\; \begin{cases} 1, & r=0 \\ \displaystyle\int_a^1 h(r-1,a,x)\,g(y-x)\,dx, & r>0,\; 0<y<1. \end{cases} \tag{3.5}$$

Finally, the recursion (3.5) will be expressed in terms of the generating function,

$$h^*(z,a,y) \;=\; \sum_{r=1}^{\infty} h(r,a,y)\,z^{r-1}, \tag{3.6}$$

of $h(r,a,y)$. To this end, multiply Eq. (3.5) by z^{r-1}, and sum over $r = 1, 2, \ldots$ to obtain the requisite integral equation

$$h^*(z,a,y) \;=\; h(1,a,y) + z\int_a^1 h^*(z,a,x)\,g(y-x)\,dx, \quad 0<y<1, \tag{3.7}$$

in the unknown generating function, $h^*(z,a,y)$.

It may be noted that (3.7) is not a Fredholm integral equation (see Tricomi (1957), Chapter 3), since the support of y is beyond the integration range; it is, in fact, of the nature of a Weiner-Hopf equation. Reduction to the Fredholm type will be accomplished in connection with Enskog's relation.

4 EXACT MATRIX SOLUTION OF THE INTEGRAL EQUATION

In this section, we derive a representation for the exact solution of the integral equation (3.7). We shall use the standard technique for Pincherle-Goursat (PG) kernels (see, e.g., Tricomi (1957), Chapter 3) to reduce (3.7) to an algebraic form. A *PG kernel* $K(x,y)$ has the form $K(x,y) = \sum_{i=1}^{n} a_i(x)\, b_i(y)$, for some finite n and some real functions $a_i(x)$ and $b_i(y)$. In our case, the kernel is, in fact, of a Hilbert type, viz., $K(x,y) = \sum_{i=1}^{\infty} a_i(x)\, b_i(y)$, but a standard truncation approach will reduce the problem to the PG type with finite n, and then we can, in principle, take the limit as $n \uparrow \infty$.

Consider now Eqs. (2.4) and (3.7), and substitute the Fourier series (2.3) evaluated at $y - x$ into Eq. (3.7), yielding

$$h^*(z,a,y) \;=\; h(1,a,y) \;+\; z \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, e^{i2\pi\nu y} \int_a^1 h^*(z,a,x)\, e^{-i2\pi\nu x}\, dx. \tag{4.1}$$

For every integer ν, define the function

$$c_\nu(z,a) \;=\; \int_a^1 h^*(z,a,x)\, e^{-i2\pi\nu x}\, dx, \tag{4.2}$$

and substitute it into Eq. (4.1), resulting in

$$h^*(z,a,y) \;=\; h(1,a,y) \;+\; z \sum_{n=-\infty}^{\infty} \tilde{f}_V(i2\pi n)\, c_n(z,a)\, e^{i2\pi n y}. \tag{4.3}$$

Now, substitute $h^*(z,a,y)$ from Eq. (4.3) into Eq. (4.2), yielding

$$\begin{aligned} c_\nu(z,a) \;&=\; \int_a^1 h(1,a,x)\, e^{-i2\pi\nu x}\, dx \\ &+\; z \sum_{n=-\infty}^{\infty} \tilde{f}_V(i2\pi n)\, c_n(z,a) \int_a^1 e^{i2\pi(n-\nu)x}\, dx. \end{aligned} \tag{4.4}$$

Next, for every integer ν, define the function

$$q_\nu(a) \;=\; \int_a^1 h(1,a,x)\, e^{-i2\pi\nu x}\, dx, \tag{4.5}$$

and for all integers ν and n, define the kernel

$$k_{\nu n}(a) \;=\; \begin{cases} (1-a)\,\tilde{f}_V(i2\pi\nu), & \nu = n \\[2ex] \tilde{f}_V(i2\pi n)\,\dfrac{1-e^{i2\pi(n-\nu)a}}{i2\pi(n-\nu)}, & \nu \neq n. \end{cases} \tag{4.6}$$

In view of Eqs. (4.5) and (4.6), Eq. (4.4) becomes

$$c_\nu(z,a) \;=\; q_\nu(a) \;+\; z \sum_{n=-\infty}^{\infty} k_{\nu n}(a)\, c_n(z,a). \tag{4.7}$$

Let $\mathbf{c}(z,a) = [c_\nu(z,a)]_{\nu=-\infty}^{\infty}$ and $\mathbf{q}(a) = [q_\nu(a)]_{\nu=-\infty}^{\infty}$ denote infinite column vectors, and let $\mathbf{K}(a) = [k_{\nu n}(a)]_{\nu,n=-\infty}^{\infty}$ denote an infinite matrix. Then, Eq. (4.7) in vector-matrix form becomes

$$\mathbf{c}(z,a) \;=\; \mathbf{q}(a) \;+\; z\,\mathbf{K}(a)\,\mathbf{c}(z,a), \tag{4.8}$$

and the formal solution of Eq. (4.8) for $\mathbf{c}(z,a)$ is

$$\mathbf{c}(z,a) \;=\; [\mathbf{I} - z\,\mathbf{K}(a)]^{-1}\,\mathbf{q}(a). \tag{4.9}$$

Since $\mathbf{K}(a)$ is already given in Eq. (4.6) in terms of known quantities, it only remains to exhibit $\mathbf{q}(a)$ in terms of known quantities. To this end, we note that Eq. (3.5) implies $h(1,a,y) \;=\; \int_a^1 g(y-x)\,dx$, whence from (2.3) and (2.4),

$$\begin{aligned} h(1,a,y) \;&=\; \sum_{n=-\infty}^{\infty} \tilde{f}_V(i2\pi n)\, e^{i2\pi ny} \int_a^1 e^{-i2\pi nx}\, dx \\ &=\; 1 \;-\; a \;+\; \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tilde{f}_V(i2\pi n)\, \frac{1-e^{i2\pi na}}{-i2\pi n}\, e^{i2\pi ny}. \end{aligned} \tag{4.10}$$

Substituting the above into (4.5), we get

$$q_\nu(a) \;=\; \begin{cases} (1-a)^2 \;+\; \displaystyle\sum_{\substack{n=-\infty \\ n\neq 0}} \tilde{f}_V(i2\pi\nu)\left(\frac{\sin(\pi n a)}{\pi n}\right)^2, & \nu = 0 \\[2ex] (1-a)\dfrac{1-e^{-i2\pi\nu a}}{-i2\pi\nu}\,[1+\tilde{f}_V(i2\pi\nu)] & \\[2ex] +\displaystyle\sum_{\substack{n=-\infty \\ 0\neq n\neq \nu}}^{\infty} \tilde{f}_V(i2\pi n)\left(\frac{1-e^{-i2\pi n a}}{-i2\pi n}\right)\left(\frac{1-e^{i2\pi(n-\nu)a}}{i2\pi(n-\nu)}\right), & \nu \neq 0. \end{cases} \tag{4.11}$$

The foregoing discussion can be summarized in

Proposition 1 *The solution of the integral equation (3.7) for $h^*(z,a,y)$ is given in terms of $h(1,a,y)$ and $c_n(z,a)$ by Eq. (4.3). Furthermore, $h(1,a,y)$ is given by Eq. (4.10), and the $c_n(z,a)$ are given by Eq. (4.9) in terms of the $q_\nu(a)$ and $k_{\nu n}(a)$. Finally, the $q_\nu(a)$ are given by Eq. (4.11), and the $k_{\nu n}$ by Eq. (4.6).* $\square$

To relate the Fredholm operator to the run probabilities of a TES process (see Section 2.2), we observe that Eq. (3.7) can be written as a Fredholm integral equation (2.7). To this end, first fix the parameter $0 < a < 1$, and then make the identifications

$$\phi(y) \;=\; h^*(z,a,y), \quad f(y) \;=\; h(1,a,y), \quad \lambda \;=\; z, \quad c \;=\; 0, \quad d \;=\; 1,$$

and

$$K(x,y) \;=\; g(y-x)\,1_a(x),$$

where $g(x)$ is given by Eq. (2.3), and $1_a(x)$ is the indicator function

$$1_a(x) \;=\; \begin{cases} 1, & x > a \\ 0, & x \leq a. \end{cases}$$

The Enskog relation (2.11) now yields the straightforward

Proposition 2 *For any real square-integrable function $\xi(y)$ on $(0,1)$,*

$$\int_0^1 \left[\xi(y) - z\,1_a(y)\int_0^1 \xi(x)\,g(y-x)\,dx\right] h^*(z,a,y)\,dy \;=\; \int_0^1 \xi(y)\,h(1,a,y)\,dy. \tag{4.12}$$

$\square$

Choosing $\xi(y) \equiv 1$ and noting that $\int_0^1 g(y-x)\,dy = 1$ due to the uniformity of the TES process $\{U_n^+\}$, Eq. (4.12) reduces to

$$\sum_{r=1}^{\infty} H(r,a)\,z^{r-1} = 1 - a - (1-z)\sum_{r=1}^{\infty} \bar{H}(r,a)\,z^{r-1}. \tag{4.13}$$

Setting $z = 1$ in Eq. (4.13) readily yields

$$\sum_{r=1}^{\infty} H(r,a) = 1 - a. \tag{4.14}$$

Furthermore, by equating coefficients in Eq. (4.13), we get the relation

$$H(r,a) = \bar{H}(r-1,a) - \bar{H}(r,a), \quad r \geq 1. \tag{4.15}$$

Note that since $H(0,a) = a$, Eq. (4.14) implies that $\bar{H}(\infty,a) = 0$ in view of Eq. (1.3) with $X_n = U_n^+$.

5 A SOKOLOV APPROXIMATION FOR $H(r,a)$

The exact solution outlined by Proposition 1 in Section 4 is of theoretical interest. However, it does not lend itself to an efficient computation, since in order to get the run probabilities, $H(r,a)$, one needs to compute first $h^*(z,a,y)$ on a (z,y) grid, and then invert and integrate out the y variable. In this section, we obtain an efficient approximation procedure for the $H(r,a)$, based on the exponential-subspace Sokolov approximation described in Section 2.4. However, we first modify the representation of the integral equation (3.7), in order to focus explicitly on the effect of autocorrelation among the U_n^+.

Consider the case when the innovation variates, V_n, have the Uniform$[0,1)$ marginal distribution. The resulting TES sequence is then i.i.d. Uniform$[0,1)$ as well, so that $g^+(y|x) \equiv 1$. The run probabilities and the associated functions become

$$\begin{aligned} H_G(r,a) &= a(1-a)^r, \quad r \geq 0, & (5.1)\\ h_G(r,a,y) &= (1-a)^r, \quad r \geq 0, \;\; 0 \leq y < 1, & (5.2)\\ h_G^*(z,a,y) &= \frac{1-a}{1-(1-a)z}, \quad 0 \leq y < 1, & (5.3) \end{aligned}$$

where the subscript G stands mnemonically for a geometric distribution of the run probabilities. Observe that Eq. (3.7) is then trivially satisfied by Eq. (5.3). If we view the special case of the i.i.d. TES process as a "reference point", it will be convenient to represent the general case in terms of an "offset" from the special case. Specifically, for the general case, we represent

$$h^*(z,a,y) \;=\; h^*_G(z,a,y) + \delta^*(z,a,y), \tag{5.4}$$

and reduce the problem of computing $h^*(z,a,y)$ to that of computing the "offset" generating function $\delta^*(z,a,y)$, as $h^*_G(z,a,y)$ is already given in (5.3). In view of the representation (5.4), Eq. (3.7) becomes

$$\delta^*(z,a,y) \;=\; \frac{h(1,a,y) - 1 + a}{1-(1-a)z} \;+\; z\int_a^1 \delta^*(z,a,x)\,g(y-x)\,dx. \tag{5.5}$$

Successive substitutions of $\delta^*(z,a,y)$ into (5.5) give rise to a solution for $\delta^*(z,a,y)$ which combines with $h^*_G(z,a,y)$ to yield an alternate representation for $h^*(z,a,y)$, given by

$$h^*(z,a,y) \;=\; \frac{h(1,a,y) + \sum_{r=2}^{\infty}[h(r,a,y) - (1-a)h(r-1,a,y)]\,z^{r-1}}{1-(1-a)z}. \tag{5.6}$$

We are now ready to apply the exponential-subspace Sokolov approximation (2.22) to $\delta^*(z,a,y)$ in the integral equation (5.5). Recall that this approximation selects the initial approximation $\phi_0(y) \;=\; \dfrac{h(1,a,y)-1+a}{1-(1-a)z} \;=\; f(y)$, which is the forcing function of (5.5). To motivate this choice, note that this forcing function is a generating function, whereas the one in the original integral equation (3.7) is not; the latter is the density $h(1,a,y)$. Since the requisite unknown function, $h^*(z,a,y)$, is a generating function, it is intuitively clear that selecting a generating function as an initial approximation is a better choice. Aposteriori, this is the reason for switching from Eq. (3.7) to the alternate formulation in Eq. (5.5). The exponential-subspace Sokolov approximation for $\delta^*(z,a,y)$ is now obtained by substituting

$$c \;=\; a, \quad d \;=\; 1, \quad \lambda \;=\; z, \quad K(x,y) \;=\; g(y-x)$$

in Eq. (2.22). Also, from Eq. (2.16), one has

$$\lambda_0^{-1} \;\simeq\; \frac{1}{1-a}\int_a^1\int_a^1 g(y-x)\,dx\,dy \;=\; \frac{\bar{H}(1,a)}{1-a} \;=\; \hat{\lambda}_0, \quad \omega \;=\; \ln(\hat{\lambda}_0).$$

With the aid of the auxiliary function

$$c(a,x) \;=\; \frac{2\omega}{e^{2\omega} - e^{2\omega a}} \int_a^1 e^{\omega y}\, g(y-x)\, dy,$$

one has the following approximation

$$\begin{aligned} h^*(z,a,y) \;\simeq\; & \frac{h(1,a,y)}{1-(1-a)z} \;+\; \frac{z}{1-(1-a)z} \int_a^1 [h(1,a,x) - 1 + a]\, g(y-x)\, dx \\ & + \frac{z^2}{1-(1-a)z} \, \frac{\displaystyle\int_a^1 [h(1,a,x) - 1 + a]\, c(a,x)\, dx}{\displaystyle 1 - z \int_a^1 e^{ax}\, c(a,x)\, dx} \int_a^1 e^{ax}\, g(y-x)\, dx. \end{aligned}$$

This approximation further simplifies to

$$\begin{aligned} h^*(z,a,y) \;\simeq\;& h(1,a,y) \;+\; h(2,a,y)\, \frac{z}{1-(1-a)z} \\ & + \frac{\displaystyle z^2 \int_a^1 [h(1,a,x) - 1 + a]\, c(a,x)\, dx}{\displaystyle [1-(1-a)z]\, [1 - z \int_a^1 e^{\omega x}\, c(a,x)\, dx]} \int_a^1 e^{\omega x}\, g(y-x)\, dx. \end{aligned}$$

In terms of partial fractions, one has

$$\begin{aligned} h^*(z,a,y) \;\simeq\;& h(1,a,y) \;+\; h(2,a,y)\, \frac{z}{1-(1-a)z} \\ & + \frac{1}{\displaystyle 1 - a - \int_a^1 e^{\omega x}\, c(a,x)\, dx} \times \\ & \left[\frac{z}{1-(1-a)z} - \frac{z}{\displaystyle 1 - z \int_a^1 e^{\omega x}\, c(a,x)\, dx} \right] \times \\ & \int_a^1 [h(1,a,x) - 1 + a]\, c(a,x)\, dx \int_a^1 e^{\omega x}\, g(y-x)\, dx. \end{aligned}$$

An approximation for $h(r,a,y)$ is now easily obtained by expanding the partial fractions into power series in z, yielding for $r > 1$

$$\begin{aligned} h(r,a,y) \;\simeq\;& h(2,a,y)(1-a)^{r-2} \\ & + \frac{\displaystyle\int_a^1 [h(1,a,x) - 1 + a]\, c(a,x)\, dx}{1 - a - C} \times \\ & \left[(1-a)^{r-2} - C^{r-2}\right] \int_a^1 e^{\omega x}\, g(y-x)\, dx, \end{aligned} \tag{5.7}$$

$$C \;=\; \int_a^1 e^{\omega x}\, c(a,x)\, dx. \tag{5.8}$$

From (3.4), one now has

$$\begin{aligned}\bar{H}(r,a) &\simeq \bar{H}(2,a)(1-a)^{r-2} \\ &+ \frac{\int_a^1 [h(1,a,x)-1+a]\, c(a,x)\, dx}{1-a-C} \times \\ &\left[(1-a)^{r-2} - C^{r-2}\right] \int_a^1 \int_a^1 e^{\omega x}\, g(y-x)\, dy\, dx. \end{aligned} \tag{5.9}$$

The foregoing discussion is summarized in

Proposition 3 *An explicit approximate solution for $\bar{H}(r,a)$ is given in Eq. (5.9). An explicit approximate solution for $H(r,a)$ is obtained from Eq. (5.9) and the relation (4.15).* □

6 CLOSED-FORM SOLUTION FOR EXPONENTIAL INNOVATIONS

When the transform of the innovation density, $\tilde{f}_V(s)$, is rational, then it is possible, in principle, to obtain the exact solution of the integral equation (3.7). Recall the function $g(x)$ in Eq. (2.3), representing the transition density of $\{U_n^+\}$. For rational $\tilde{f}_V(s)$, the function $g(x)$ becomes an exponential polynomial, so that a differential operator may be found to eliminate the integration. This will replace the integral equation (3.7) by a time-invariant differential equation. We illustrate this procedure for the exponential innovation density

$$f_V(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{6.1}$$

with Laplace transform

$$\tilde{f}_V(s) = \frac{\lambda}{\lambda + s}. \tag{6.2}$$

Substituting Eq. (6.2) in Eq. (2.3), yields

$$g(x) = \sum_{\nu=-\infty}^{\infty} \frac{\lambda}{\lambda + i2\pi\nu}\, e^{i2\pi\nu x} = \frac{\lambda}{1-e^{-\lambda}}\, e^{-\lambda\langle x\rangle}. \tag{6.3}$$

The integral equation (3.7) takes the form

$$\begin{aligned} h^*(z,a,y) \;=&\; \\ & h(1,a,y) \;+\; \frac{z\lambda}{1-e^{-\lambda}} \int_a^1 h^*(z,a,x)\, e^{-\lambda\langle y-x\rangle}\, dx, \quad a<y<1, \end{aligned} \tag{6.4}$$

where the restriction $a < y < 1$ is meant to indicate that the solution will be studied only in the interval $(a, 1)$. This does not incur any loss of generality, since the relation (4.13) will be used to compute the solution for $y \in [0, a]$. Rewriting Eq. (6.4) in the form

$$\begin{aligned} h^*(z,a,y) \;&=\; h(1,a,y) \\ &+\; \frac{z\lambda}{1-e^{-\lambda}} \int_a^y h^*(z,a,x)\, e^{-\lambda(y-x)}\, dx \\ &+\; \frac{z\lambda}{1-e^{-\lambda}} \int_y^1 h^*(z,a,x)\, e^{-\lambda(1+y-x)}\, dx, \end{aligned} \tag{6.5}$$

and differentiating both sides of Eq. (6.5) with respect to y, yields the ordinary differential equation

$$\begin{aligned} \frac{dh^*(z,a,y)}{dy} \;&=\; \frac{dh(1,a,y)}{dy} \;+\; \lambda z\, h^*(z,a,y) \\ &-\; \lambda\, \frac{z\lambda}{1-e^{-\lambda}} \int_a^y h^*(z,a,x)\, e^{-\lambda(y-x)}\, dx \\ &-\; \lambda\, \frac{z\lambda}{1-e^{-\lambda}} \int_y^1 h^*(z,a,x)\, e^{-\lambda(1+y-x)}\, dx. \end{aligned} \tag{6.6}$$

With the aid of Eq. (6.5), we can simplify Eq. (6.6) to

$$\frac{dh^*(z,a,y)}{dy} + \lambda(1-z)h^*(z,a,y) \;=\; \frac{dh(1,a,y)}{dy} + \lambda h(1,a,y). \tag{6.7}$$

But setting $r = 1$ in Eq. (3.5) and in view of Eq. (3.2), we deduce $h(1, a, y) = \int_a^1 g(y-x)dx$, which on differentiation with the aid of Eq. (2.5) yields

$$\frac{dh(1,a,y)}{dy} \;+\; \lambda\, h(1,a,y) \;=\; \lambda.$$

Consequently, we can further simplify Eq. (6.7) to

$$\frac{dh^*(z,a,y)}{dy} + \lambda(1-z)\, h^*(z,a,y) \;=\; \lambda, \quad a<y<1. \tag{6.8}$$

The general solution of the ordinary differential equation (6.8) is

$$h^*(z,a,y) \;=\; \frac{1}{1-z} \;+\; s(z,a)\, e^{-\lambda(1-z)y}. \tag{6.9}$$

The determination of $s(z,a)$ is accomplished by substitution of Eq. (6.9) into the integral equation (6.4). This finally yields

$$h^*(z,a,y) \;=\; \frac{1}{1-z}\left[1 - \frac{e^{\lambda a}-1}{e^{\lambda a z} - e^{-\lambda(1-z)}}\, e^{-\lambda(1-z)y}\right]. \tag{6.10}$$

To obtain the requisite quantities, $H(r,a)$, we first observe that from Eq. (3.4),

$$\sum_{r=1}^{\infty} \bar{H}(r,a)\, z^{r-1} \;=\; \int_a^1 h^*(z,a,y)\, dy \;=$$
$$\frac{1-a}{1-z} - \frac{1}{1-z}\, \frac{e^{\lambda a}-1}{e^{\lambda a z} - e^{-\lambda(1-z)}}\, \frac{e^{-\lambda(1-z)a} - e^{-\lambda(1-z)}}{\lambda\,(1-z)}. \tag{6.11}$$

Use of the relation (4.13) now yields

$$\sum_{r=1}^{\infty} H(r,a)\, z^{r-1} \;=\; \frac{e^{\lambda a}-1}{e^{\lambda a z} - e^{-\lambda(1-z)}}\, \frac{e^{-\lambda(1-z)a} - e^{-\lambda(1-z)}}{\lambda\,(1-z)}. \tag{6.12}$$

The eigenvalues, $\{z_k\}_{-\infty}^{\infty}$, of the integral equation (3.7) are poles of the solution as a function of z; that is, they are zeros of the function $e(z) \;=\; e^{\lambda a z} - e^{-\lambda\,(1-z)}$. Therefore,

$$z_k \;=\; \frac{1}{1-a} \;+\; i\, \frac{2\pi k}{\lambda\,(1-a)}. \tag{6.13}$$

Each eigenfunction, $\phi_k(y)$, belonging to the respective eigenvalue, z_k, solves the homogeneous differential equation

$$\frac{d\phi_k(y)}{dy} \;+\; \lambda\,(1-z_k)\,\phi_k(y) \;=\; 0, \tag{6.14}$$

namely,

$$\phi_k(y) \;=\; e^{-\lambda(1-z_k)y}. \tag{6.15}$$

It follows that the dominant eigenvalue (absolutely lowest) $z_0 \;=\; \frac{1}{1-a}$ is real; interestingly, z_0 is independent of λ.

7 NUMERICAL EXAMPLES

To obtain an indication of the efficacy of our approach, we compared solutions of the run-probabilities problem for a uniform TES process with exponential innovation densities. Tables 1 and 2 show the results of the comparison for level $a = 0.3$, and run lengths $r = 1, 2, \ldots, 10$. In each case, we compared statistics based on a Monte Carlo simulation of 10^6 observations to the exponential-subspace Sokolov approximation of Section 5, and then again to the exact closed-form solution of Section 6.

Table 1 displays the results for an exponential innovation with parameter $\lambda = 1.0$, and Table 2 displays the same for $\lambda = 4.0$. Note that the TES process corresponding to the latter case is more positively autocorrelated, since the corresponding innovation density decays more quickly. As a consequence of the higher positive autocorrelations, the run probabilities in Table 2 decay more slowly than their counterparts of Table 1. The agreement in Tables 1 and 2 between the analytical approximation and its Monte Carlo estimate and exact solution counterparts is excellent. Overall, the results exhibited support the efficacy of the Sokolov approximation technique used here. Evidently, the exponential subspace, M_E, in the Sokolov approximation is a good choice, corresponding to the "intrinsic coordinates" of the problem at hand.

One of the goals of this paper is to draw workers' attention to the Sokolov approximation technique. Admittedly, the goodness of a Sokolov-type approximation depends crucially on the choice of the functional subspace, M, and the initial approximation iterate, $\phi_0(y)$. Both of these choices are heuristic in nature, as no constructive procedure is known for making a good selection as a function of the posed problem. Thus, practitioners of the Sokolov approximation technique employ largely Art rather than Science, and must rely to a large extent on prior experience, lucky insight or just plain trial and error. To illustrate this point, we mention, in passing, that the exponential subspace, M_E, was not our first choice; an earlier choice of a Sokolov subspace, spanned by a constant function, had been tried unsuccessfully. In spite of the heuristic trial-and-error element involved in applying Sokolov approximations, we suggest that for many hard computational problems, this technique can achieve excellent results, and is well-worth a hard try.

Run Length r	Sokolov Analytical Approximation	Monte Carlo Simulation Statistics	Exact Solution of Integral Equation
1	2.0641 E-1	2.0662 E-1	2.0641 E-1
2	1.4789 E-1	1.4813 E-1	1.4797 E-1
3	1.0370 E-1	1.0377 E-1	1.0371 E-1
4	7.2593 E-2	7.2369 E-2	7.2575 E-2
5	5.0818 E-2	5.0831 E-2	5.0800 E-2
6	3.5574 E-2	3.5743 E-2	3.5560 E-2
7	2.4903 E-2	2.4885 E-2	2.4892 E-2
8	1.7433 E-2	1.7402 E-2	1.7425 E-2
9	1.2204 E-2	1.2268 E-2	1.2197 E-2
10	8.5430 E-3	8.4860 E-3	8.5380 E-3

Table 1: Comparison of Computations of $H(r, a)$ at Level $a = 0.3$, for a Uniform TES Process with Exponential Innovation Density with Rate $\lambda = 1.0$

Run Length r	Sokolov Analytical Approximation	Monte Carlo Simulation Statistics	Exact Solution of Integral Equation
1	1.6714 E-1	1.6767 E-1	1.6714 E-1
2	1.4520 E-1	1.4591 E-1	1.4557 E-1
3	1.0721 E-1	1.1436 E-2	1.1425 E-1
4	7.7225 E-2	8.2661 E-2	8.2847 E-2
5	5.5715 E-2	5.7611 E-2	5.7714 E-2
6	4.0261 E-2	3.9705 E-2	3.9888 E-2
7	2.9141 E-2	2.7517 E-2	2.7745 E-2
8	2.1127 E-2	1.9350 E-2	1.9422 E-2
9	1.5342 E-2	1.3570 E-2	1.3620 E-2
10	1.1160 E-2	9.4960 E-3	9.5442 E-3

Table 2: Comparison of Computations of $H(r, a)$ at Level $a = 0.3$, for a Uniform TES Process with Exponential Innovation Density with Rate $\lambda = 4.0$

The reader is referred to Jagerman (1987) for an application of the Sokolov approximation technique to the waiting time distribution in GI/G/1 queues.

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