3 INTEGRAL EQUATIONS FOR MARKOVIAN TRAFFIC

Recall that by assumption, $t = 0$ is an arrival point, $-X_0$ is the previous arrival point and X_1 the next arrival point. Denoting $q_n(t|x) = P\{N(t) = n|X_0 = x\}$, define the generating function $G(z,t|x) = E[z^{N(t)}|X_0 = x] = \sum_{n=0}^{\infty} q_n(t|x)\, z^n$. We now proceed to derive an integral equation for $G(z,t|x)$. Noting the sample path relation

$$z^{N(t)} \;=\; \begin{cases} 1, & \text{on } \{X_1 > t\} \\ z\, z^{N(t-X_1)}, & \text{on } \{X_1 \le t\} \end{cases} \tag{3.1}$$

and recalling that $f_1(y|x)$ is the 1-step transition density of $\{X_n\}$, it follows from (3.1) that

$$E\left[1_{\{X_1>t\}}|X_0 = x\right] \;=\; \int_t^{\infty} f_1(y|x)\,dy$$

$$E\left[z^{1+N(t-X_1)}1_{\{X_1\le t\}}|X_0 = x\right] \;=\; z\int_0^t G(z,t-y|y)\,f_1(y|x)\,dy.$$

Hence, the required integral equation for $G(z,t|y)$ is

$$G(z,t|x) \;=\; \int_t^{\infty} f_1(y|x)\,dy \;+\; z\int_0^t G(z,t-y|y)\,f_1(y|x)\,dy. \tag{3.2}$$

Unfortunately, (3.2) is neither of Volterra nor Fredolm type. For later applications it will be more convenient to transform it into a Fredholm-type integral equation. To do that, observe that the conditions of Lemma 1 are satisfied for $f_1(y|x)$ and $G(z,t|y)$ in the set $\{z:\; |z| \le 1\}$. Thus, taking Laplace transforms in (3.2) yields

$$\tilde{G}(z,s|x) \;=\; \frac{1-\tilde{f}_1(s|x)}{s} \;+\; z\int_0^{\infty} e^{-sy}\,\tilde{G}(z,s|y)\,f_1(y|x)\,dy, \quad s > 0. \tag{3.3}$$

The integral equation for the Laplace transform $\tilde{M}(s|x)$ of the conditional expectation $M(t|x) = E[N(t)|x]$ is obtained from the integral equation (3.3) using

$$\tilde{M}(s|x) \;=\; \frac{\partial}{\partial z}\tilde{G}(z,s|x)|_{z=1} \;=\; \frac{1}{s}\,\tilde{f}_1(s|x) \;+\; \int_0^{\infty} e^{-sy}\,\tilde{M}(s|y)\,f_1(y|x)\,dy, \tag{3.4}$$

justified by the uniform convergence of $\dfrac{\partial}{\partial z}\tilde{G}(z,s|x)$ in the set $\{z:\; |z| \le 1\}$, by appeal to Theorem 1.

Finally, we obtain an integral equation for the Laplace transform of the generating function $L(z,t|x) = \sum_{n=1}^{\infty} p_n(t|x)\, z^{n-1}$, where $p_n(t|x) = \dfrac{\partial}{\partial t}P\{S_n \le t|X_0 = x\}$. Since

$$P\{S_n > t|X_0 = x\} \;=\; P\{N(t) < n|X_0 = x\} \;=\; \sum_{k=0}^{n-1} q_k(t|x),$$

we can write $p_n(t|x) = -\dfrac{\partial}{\partial t}\displaystyle\sum_{k=0}^{n-1} q_k(t|x)$, whence

$$L(z,t|x) \;=\; -\frac{\partial}{\partial t}\sum_{n=1}^{\infty} z^{n-1}\sum_{k=0}^{n-1} q_k(t|x) \;=\; -\frac{\partial}{\partial t}\,\frac{G(z,t|x)}{1-z}. \tag{3.5}$$

Applying Laplace tranforms to (3.5) yields the relation

$$\tilde{L}(z,s|x) \;=\; \frac{1 - s\,\tilde{G}(z,s|x)}{1-z},$$

and applying (3.3) to the relation above results in the integral equation

$$\tilde{L}(z,s|x) \;=\; \tilde{f}_1(s|x) \;+\; z\int_0^\infty e^{-sy}\,\tilde{L}(z,s|y)\,f_1(y|x)\,dy \tag{3.6}$$

for $\tilde{L}(z,s|x)$. It is interesting to note that

$$\tilde{M}(s|x) \;=\; \frac{1}{s}\,\tilde{L}(1,s|x), \tag{3.7}$$

and this relation extends the known formula for the transform of the renewal density in renewal theory (Cox [1]).

The three equations, (3.3), (3.4) and (3.6), all have the generic form (2.1). We are now in a position to use the Fredholm-type integral equation theory, developed in Section 2, to solve the integral equations (3.3), (3.4) and (3.6).

For the integral equation (3.3), the forcing term, $h_G(x) = \dfrac{1-\tilde{f}_1(s|x)}{s}$, is assumed to belong to $L^2([0,\infty))$, or to be bounded on $[0,\infty]$ and summable over every finite interval. The solution of (3.3) is

$$\tilde{G}(z,s|x) \;=\; \frac{1-\tilde{f}_1(s|x)}{s} \;+\; \sum_{n=1}^{\infty} z^n\,\mathcal{K}_s^n\!\left[\frac{1-\tilde{f}_1(s|x)}{s}\right]. \tag{3.8}$$

Since the circle of convergence of the Neumann series (2.7) includes $|z| = 1$, the integral equation (3.4) for $\tilde{M}(s|x)$ is established, as noted earlier, as well as its existence. For the integral equation (3.4), the forcing term, $h_M(x) = \dfrac{\tilde{f}_1(s|x)}{s}$, is also assumed to belong to $L^2([0,\infty))$, or to be bounded on $[0,\infty]$ and summable over every finite interval. The solution of (3.4) is

$$\tilde{M}(s|x) \;=\; \frac{1}{s}\,\tilde{f}_1(s|x) \;+\; \frac{1}{s}\sum_{n=1}^{\infty}\mathcal{K}_s^n[\tilde{f}_1(s|x)] \;=\; \frac{1}{s}\,\tilde{f}_1(s|x) \;+\; \frac{1}{s}\,R_s(1)\,\tilde{f}_1(s|x). \tag{3.9}$$

For the integral equation (3.6), the forcing term, $h_L(x) = \tilde{f}_1(s|x)$, is again assumed to belong to $L^2([0,\infty))$, or to be bounded on $[0,\infty]$ and summable over every finite interval. The solution of (3.6) is

$$\tilde{L}(z,s|x) \;=\; \tilde{f}_1(s|x) \;+\; \sum_{n=1}^{\infty} z^n\,\mathcal{K}_s^n[\tilde{f}_1(s|x)]. \tag{3.10}$$

The coefficient of z^n in (3.8) is the transform $\tilde{q}_n(s|x)$, whence

$$\tilde{q}_n(s|x) \;=\; \mathcal{K}_s^n\!\left[\frac{1-\tilde{f}_1(s|x)}{s}\right] \;=\; \int_0^\infty \frac{1-\tilde{f}_1(s|y)}{s}\,K_{s,n}(x,y)\,dy, \quad n \ge 1. \tag{3.11}$$

Finally, The coefficient of z^n in (3.10) is the transform $\tilde{p}_{n+1}(s|x)$, whence

$$\tilde{p}_n(s|x) \;=\; \mathcal{K}_s^{n-1}[\tilde{f}_1(s|x)] \;=\; \int_0^\infty \tilde{f}_1(s|y)\,K_{s,n-1}(x,y)\,dy, \quad n \ge 1. \tag{3.12}$$

4 SPECIALIZATION TO TES$^+$ TRAFFIC PROCESSES

We now proceed to specialize the discussion to a TES$^+$ arrival process $\{X_n^+\}$, with innovation density f_V. From (1.5), the transition density, $g_U^+(v|u)$, of the uniform background TES$^+$ process, $\{U_n^+\}$, has the representation

$$g_U^+(v|u) \;=\; g_U(v-u),$$

in which the function g_U on the right-hand side is given by

$$g_U(w) \;=\; \sum_{n=-\infty}^{\infty} f_V(w+n) \;=\; \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, e^{i2\pi\nu w}, \tag{4.1}$$

where the second equality follows from the Poisson summation formula (Lighthill [12]).

Next, we rewrite the integral equations from Section 3 for their $\{X_n^+\}$ counterparts $\tilde{G}_X^+(z,s|x)$, $\tilde{M}_X^+(s|x)$ and $\tilde{L}_X^+(z,s|x)$ in terms of the function $g_U(w)$. This has the important advantage of transforming the infinite integration range to the compact set $[0,1]$. To this end, observe that the probability element $f_1^+(y|x)dy$, with $x = D(u)$ and $y = D(v)$, is transformed to

$$f_1^+(y|x)\,dy \;=\; f_1^+(D(v)|D(u))\,D'(v)\,dv \;=\; g_U(v-u)\,dv. \tag{4.2}$$

Consequently, the integral equation (3.3) is transformed to

$$\tilde{G}_X^+(z,s|D(u)) \;=\; \frac{1-\tilde{f}_1^+(s|D(u))}{s} + z\int_0^1 \tilde{G}_X^+(z,s|D(v))\,e^{-sD(v)}\,g_U(v-u)\,dv, \tag{4.3}$$

the integral equation (3.4) is transformed to

$$\tilde{M}_X^+(s|D(u)) \;=\; \frac{1}{s}\,\tilde{f}_1^+(s|D(u)) \;+\; \int_0^1 \tilde{M}_X^+(s|D(v))\,e^{-sD(v)}\,g_U(v-u)dv, \tag{4.4}$$

and the integral equation (3.6) is transformed to

$$\tilde{L}_X^+(z,s|D(u)) \;=\; \tilde{f}_1^+(s|D(u)) \;+\; z\int_0^1 \tilde{L}_X^+(z,s|D(v))\,e^{-sD(v)}\,g_U(v-u)\,dv. \tag{4.5}$$

For each u, the function $\tilde{M}_X^+(s|D(u))$ is analytic in the plane $\{s:\ Re[s]>0\}$ with a pole at $s=0$. The contribution of this pole (to be used in Section 7) is given in the next theorem.

Theorem 2 *For any TES$^+$ process $\{X_n^+\}$ of the form (1.3), the asymptotic expansion of $\tilde{M}_X^+(s|D(u))$ at 0 is given, for each u, by*

$$\tilde{M}_X^+(s|D(u)) \;\sim\; \frac{\lambda_X^+}{s^2} \;+\; \frac{b_X^+(u)}{s}, \quad s\to 0+, \tag{4.6}$$

where $\lambda_X^+ = 1/E[X_n]$, and

$$b_X^+(u) \;=\; b_0^+ - \lambda_X^+ \sum_{\nu=-\infty}^{\infty} \frac{\tilde{f}_V(-i2\pi\nu)}{1-\tilde{f}_V(-i2\pi\nu)}\,\tilde{D}(i2\pi\nu)\,e^{i2\pi\nu u}, \tag{4.7}$$

for some constant b_0^+, to be determined in Section 7.

Proof The asymptotic analysis will be carried out, postulating the expansion (4.6) and substituting it into (4.4). Accordingly,

$$\frac{\lambda_X^+}{s^2} + \frac{b_X^+(u)}{s} = \frac{1}{s} + \int_0^1 \left(\frac{\lambda_X^+}{s^2} + \frac{b_X^+(v)}{s} \right) [1 - s\, D(v)]\; g(v-u)\, dv, \tag{4.8}$$

in which only the relevant powers of s have been retained, and the first term on the right-hand side, $1/s$, arises from $\tilde{f}_1^+(0|D(u)) \equiv 1$ by expanding it in powers of s around 0. Equating the coefficients of $1/s^2$ in (4.8) yields

$$\lambda_X^+ = \lambda_X^+ \int_0^1 g(v-u)\, dv,$$

which is clearly satisfied since $\int_0^1 g(v-u)\, dv \equiv 1$. Equating the coefficients of $1/s$ in (4.8) yields the following integral equation for $b_X^+(u)$,

$$b_X^+(u) = 1 - \lambda_X^+ \int_0^1 D(v)\, g(v-u)\, dv + \int_0^1 b_X^+(v)\, g(v-u)\, dv.$$

To solve the integral equation above for b_X^+, substitute the Fourier series representation of $g(v-u)$ from (4.1), which gives

$$b_X^+(u) = 1 - \lambda_X^+ \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, \tilde{D}(-i2\pi\nu)\, e^{-i2\pi\nu u} + \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, \tilde{b}_X^+(-i2\pi\nu)\, e^{-i2\pi\nu u}.$$

To put the Fourier series above in standard form, we use complex conjugates to obtain

$$b_X^+(u) = 1 - \lambda_X^+ \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, \tilde{D}(i2\pi\nu)\, e^{i2\pi\nu u} + \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, \tilde{b}_X^+(i2\pi\nu)\, e^{i2\pi\nu u}. \tag{4.9}$$

On the other hand, since

$$b_X^+(u) = \sum_{\nu=-\infty}^{\infty} \tilde{b}_X^+(i2\pi\nu)\, e^{i2\pi\nu u}, \tag{4.10}$$

one may equate coefficients in the representations (4.9) and (4.10) and deduce that

$$\tilde{b}_X^+(i2\pi\nu) = \begin{cases} 1 - \lambda_X^+ \tilde{D}(0) + \tilde{b}_X^+(0), & \nu = 0 \\[2ex] -\lambda_X^+ \tilde{f}_V(-i2\pi\nu)\, \tilde{D}(i2\pi\nu) + \tilde{f}_V(-i2\pi\nu)\, \tilde{b}_X^+(i2\pi\nu), & \nu \neq 0 \end{cases}$$

Since $\tilde{D}(0) = 1/\lambda_X^+$, the equation for $\tilde{b}_X^+(0)$ is consistent but does not determine $\tilde{b}_X^+(0)$. This constant will be determined later on in Section 7. But for $\nu \neq 0$,

$$\tilde{b}_X^+(i2\pi\nu) = -\lambda_X^+ \frac{\tilde{f}_V(-i2\pi\nu)}{1 - \tilde{f}_V(-i2\pi\nu)}\, \tilde{D}(i2\pi\nu).$$

The theorem follows, since the postulated asymptotic expansion is consistent. □

Since $M(t|D(u))$ is monotone increasing in t (for fixed $D(u)$), a real Tauberian theorem [15] may be used to obtain the following asymptotic expansion at infinity,

$$M(t|D(u)) \sim \lambda t, \quad t \to \infty.$$

One may also expect

$$M(t|D(u)) \sim \lambda t \ + \ b_X^+(u),$$

although this does not directly follow from the Tauberian theorem.

The three equations (4.3)–(4.5), all have the generic form

$$\varphi(u) \ = \ h(u) \ + \ z \int_0^1 \varphi(v)\, T_s(u,v)\, dv, \tag{4.11}$$

where the kernel $T_s(u,v)$ is given by

$$T_s(u,v) \ = \ e^{-sD(v)}\, g_U(v-u). \tag{4.12}$$

Thus, the Fredholm-type integral equation (4.11) is a special case of the integral equation (2.1), and the kernel $T_s(u,v)$ in (4.12) is a special case of the kernel $K_s(x,y)$ of (2.2). In conformance with the notational conventions of Section 2, the associated operator $\mathcal{T}_s$, $s > 0$, on $L^2([0,\infty))$ is

$$\mathcal{T}_s[f(u)] \ = \ \int_0^1 e^{-sD(v)} g_U(v-u)\, f(v)\, dv, \quad f(u) \in L^2([0,1)).$$

The iterated kernels $T_{s,n}(u,v)$ take the form

$$T_{s,n}(u,v) \ = \ \begin{cases} T_s(u,v), & n = 1 \\ \displaystyle\int_0^1 e^{-sD(v)}\, g_U(v-w)\, T_{s,n-1}(w,u)\, dw, & n > 1 \end{cases}$$

Since the theory outlined in Section 2 for Fredholm-type integral equations holds for $K_s = T_s$ as a special case, we may apply the solutions as well to the special case of TES$^+$ processes.

From (3.8), the solution of the integral equation (4.3) is

$$\tilde{G}_X^+(z,s|D(u)) \ = \ \frac{1-\tilde{f}_1^+(s|D(u))}{s} \ + \ \sum_{n=1}^{\infty} z^n\, \mathcal{T}_s^n\left[\frac{1-\tilde{f}_1^+(s|D(u))}{s}\right], \tag{4.13}$$

and from (3.11),

$$\begin{aligned} \tilde{q}_n^+(s|D(u) \ &= \ \mathcal{T}_s^n\left[\frac{1-\tilde{f}_1^+(s|D(u))}{s}\right] \\ &= \ \frac{1}{s}\int_0^1 [1-\tilde{f}_1^+(s|D(v))]\, T_{s,n}(u,v)\, dv, \quad n \geq 1. \end{aligned} \tag{4.14}$$

From (3.9), the solution of the integral equation (4.4) is

$$\begin{aligned}\tilde{M}_X^+(s|D(u)) &= \frac{1}{s}\,\tilde{f}_1^+(s|D(u)) \;+\; \frac{1}{s}\sum_{n=1}^{\infty}\mathcal{T}_s^n[\tilde{f}_1^+(s|D(u))]\\ &= \frac{1}{s}\,\tilde{f}_1^+(s|D(u)) \;+\; \frac{1}{s}\,R_s^+(1)\,\tilde{f}_1^+(s|D(u)),\end{aligned}\tag{4.15}$$

where R_s^+ is the resolvent (2.8), corresponding to the TES kernel $T_s(u,v)$.

From (3.10), the solution of the integral equation (4.5) is

$$\tilde{L}_X^+(z,s|D(u)) \;=\; \tilde{f}_1^+(s|D(u)) \;+\; \sum_{n=1}^{\infty} z^n\,\mathcal{T}_s^n[\tilde{f}_1^+(s|D(u))],\tag{4.16}$$

and from (3.12),

$$\tilde{p}_n^+(s|D(u)) \;=\; \mathcal{T}_s^{n-1}[\tilde{f}_1^+(s|D(u))] \;=\; \int_0^1 \tilde{f}_1^+(s|D(u))\,T_{s,n-1}(u,v)\,dv, \quad n\geq 1.\tag{4.17}$$

It is interesting to compare the structure of the integral equations (4.3), (4.4) and (4.5) and their respective solutions (4.13), (4.15) and (4.16) above (when the interarrival process is a Markovian TES$^+$ sequence) with the renewal case (when the interarrival process is iid, implying $\tilde{f}_1^+(s|x) \equiv \tilde{f}_X^+(s)$). The renewal case is just a special TES$^+$ process with $g_U^+(v|u) = g_U(v-u) \equiv 1$. From standard renewal theory or from the respective integral equations (4.3), (4.4) and (4.5), the corresponding transforms are

$$\begin{aligned}\tilde{G}_r(z,s) &= \frac{1}{s}\,\frac{1-\tilde{f}_X^+(s)}{1-z\,\tilde{f}_X^+(s)}\\ \tilde{M}_r(s) &= \frac{1}{s}\,\frac{\tilde{f}_X^+(s)}{1-\tilde{f}_X^+(s)}\\ \tilde{L}_r(z,s) &= \frac{\tilde{f}_X^+(s)}{1-z\,\tilde{f}_X^+(s)}\end{aligned}$$

For any $g_U(w)$, let $\tilde{G}_d(z,s,u) = \tilde{G}_X^+(z,r|D(u)) - \tilde{G}_r(z,s)$, $\tilde{M}_d(s,u) = \tilde{M}_X^+(s|D(u)) - \tilde{M}_r(s)$, and $\tilde{L}_d(z,s,u) = \tilde{L}_X^+(z,s|D(u)) - \tilde{L}_r(z,s)$ be the respective deviation functions from the renewal case. Each of the deviation functions satisfies an integral equation as follows: $\tilde{G}_d(z,s,u)$ satisfies

$$\tilde{G}_d(z,s,u) \;=\; \frac{1-\tilde{f}_1^+(s|D(u))}{s} \;-\; \tilde{G}_r(z,s)\,[1-z\,\tilde{f}_1^+(s|D(u))] \;+\; z\,\mathcal{T}_s[\tilde{G}_d(z,s,u)],$$

$\tilde{M}_d(s,u)$ satisfies

$$\tilde{M}_d(s,u) \;=\; \frac{1}{s}\,\tilde{f}_1^+(s|D(u)) \;-\; \tilde{M}_r(s)\,[1-\tilde{f}_1^+(s|D(u))] \;+\; \mathcal{T}_s[\tilde{M}_d(s,u)],$$

and $\tilde{L}_d(z,s,u)$ satisfies

$$\tilde{L}_d(z,s,u) \;=\; \tilde{f}_1^+(s|D(u)) \;-\; \tilde{L}_r(z,s)\,[1-z\,\tilde{f}_1^+(s|D(u))] \;+\; z\,\mathcal{T}_s[\tilde{L}_d(z,s,u)].$$

For TES$^+$ processes which are approximately renewal, one would expect the corresponding Neumann expansions (2.7) for $\tilde{G}_d$, $\tilde{M}_d$ and $\tilde{L}_d$ to provide good approximations. This aspect of the integral equations may be used to construct analytical approximations for the solutions.

5 A MATRIC FORM OF SOLUTION FOR TES$^+$ TRAFFIC PROCESSES

The Fourier series representation of $g_U(w)$ in (4.1) may be used to construct a matric solution for the integral equation (4.11). Following Tricomi [14] to this end, substitute (4.1) for $g_U(w)$ in (4.11), yielding

$$\varphi(u) \;=\; h(u) \;+\; z\int_0^1 \varphi(v)\, e^{-sD(v)} \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, e^{i2\pi\nu(v-u)}\, dv.$$

Next, write

$$\begin{aligned}\varphi(u) \;&=\; h(u) \;+\; z\sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, e^{-i2\pi\nu u}\int_0^1 \varphi(v)\, e^{-sD(v)\,+\,i2\pi\nu v}\, dv \\ &=\; h(u) \;+\; z\sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, e^{i2\pi\nu u}\int_0^1 \varphi(v)\, e^{-sD(v)\,-\,i2\pi\nu v}\, dv, \qquad (5.1)\end{aligned}$$

where Parseval's equality (Hardy and Rogosinski [6]) justifies the interchange of integration and summation in the first equality, and complex conjugation justifies the second equality. Denoting, for integer μ,

$$c_\mu(s) \;=\; \int_0^1 \varphi(v)\, e^{-sD(v)\,-\,i2\pi\mu v}\, dv \qquad (5.2)$$

to be the Fourier coefficients of $\varphi(v)\, e^{-sD(v)}$, we can rewrite (5.1) as

$$\varphi(u) \;=\; h(u) \;+\; z\sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, c_\nu(s)\, e^{i2\pi\nu u}. \qquad (5.3)$$

Equation (5.3) is a solution for $\varphi(u)$ in terms of $(\ldots, c_{-1}(s), c_0(s), c_1(s), \ldots)$, which is an unknown vector with components $c_\mu(s)$. In order to determine this unknown vector, substitute (5.3) into (5.2). This yields

$$\begin{aligned}c_\mu(s) \;&=\; \int_0^1 h(v)\, e^{-sD(v)\,-\,i2\pi\mu v}\, dv \\ &\quad+\; z\int_0^1 e^{-sD(v)\,-\,i2\pi\mu v} \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, c_\nu(s)\, e^{i2\pi\nu v}\, dv \\ &=\; \int_0^1 h(v)\, e^{-sD(v)\,-\,i2\pi\mu v}\, dv \\ &\quad+\; z\sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, c_\nu(s)\int_0^1 e^{-sD(v)\,+\,i2\pi(\nu-\mu)v}\, dv, \qquad (5.4)\end{aligned}$$

where Parseval's formula again justifies the interchange of integration and summation. Let $\mathbf{c}(s)$ be the vector with components $c_\mu(s)$, and let $\mathbf{h}(s)$ be the vector with components $h_\mu(s) = \int_0^1 h(v)\, e^{-sD(v)\,-\,i2\pi\mu v}\, dv$. Finally, let $\mathbf{M}(s)$ be the matrix with components $M_{\mu,\nu}(s) = \tilde{f}_V(-i2\pi\nu)\int_0^1 e^{-sD(v)\,+\,i2\pi(\nu-\mu)v}\, dv$. Equation (5.4) now takes the form

$$c_\mu(s) \;=\; h_\mu(s) \;+\; z\sum_{\nu=-\infty}^{\infty} M_{\mu,\nu}(s)\, c_\nu(s),$$

which can be written in matric form as

$$\mathbf{c}(s) \;=\; \mathbf{h}(s) \;+\; z\,\mathbf{M}(s)\,\mathbf{c}(s).$$

The solution of the matric equation above is

$$\mathbf{c}(s) \;=\; [\mathbf{I} - z\,\mathbf{M}(s)]^{-1}\,\mathbf{h}(s), \tag{5.5}$$

where $\mathbf{I}$ is the (infinite-dimensional) identity matrix. The solution (5.5) may be effectuated, in practice, by using an $n \times n$ submatrix extracted from $\mathbf{M}(s)$ by symmetrical truncation. This is the same as symmetrically truncating the Fourier series (4.1) for $g_U(w)$.

6 EXAMPLE: TES$^+$ PROCESSES WITH EXPONENTIAL INNOVATIONS

When the transform $\tilde{f}_V(s)$ of the innovation density is rational then, in principle, it is possible to obtain the exact solution for the integral equation (4.11). In that case, $g_U(w)$ has the form of an exponential polynomial, so that a differential operator may be found to eliminate the integration; this will replace the integral equation by a differential equation. Difficulties still remain, however, since the differential equation will have variable coefficients.

In this section, we illustrate this procedure for the exponential innovation density $f_V(x) = \lambda e^{-\lambda x},\ x > 0$, with its Laplace transform $\tilde{f}_V(s) = \dfrac{\lambda}{\lambda + s}$. From (4.1), the corresponding density $g_U(w)$ is given by

$$g_U(w) \;=\; \sum_{\nu=-\infty}^{\infty} \frac{\lambda}{\lambda + i2\pi\nu}\, e^{i2\pi\nu w} \;=\; \frac{\lambda}{1 - e^{-\lambda}}\, e^{-\lambda\langle w\rangle}, \tag{6.1}$$

and the corresponding transformed transition density is

$$\tilde{f}_1^{+}(s|D(u)) \;=\; \frac{\lambda}{1 - e^{-\lambda}} \int_0^1 e^{-sD(v) - \lambda\langle v-u\rangle}\, dv. \tag{6.2}$$

Putting (6.1) into the integral equation (4.11) yields

$$\begin{aligned}
\varphi_{s,z}(u) \;&=\; h(u) \;+\; z\,\frac{\lambda}{1 - e^{-\lambda}} \int_0^1 \varphi_{s,z}(v)\, e^{-sD(v) - \lambda\langle v-u\rangle}\, dv \\
&=\; h(u) \;+\; z\,\frac{\lambda}{1 - e^{-\lambda}} \int_0^u \varphi_{s,z}(v)\, e^{-\lambda - sD(v) - \lambda v + \lambda u}\, dv \\
&\qquad +\; z\,\frac{\lambda}{1 - e^{-\lambda}} \int_u^1 \varphi_{s,z}(v)\, e^{-sD(v) - \lambda v + \lambda u}\, dv.
\end{aligned} \tag{6.3}$$

After differentiating (6.3) with respect to u, we get

$$\varphi'_{s,z}(u) \;=\; h'(u) \;+\; z\,\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\,\varphi_{s,z}(u)\,e^{-sD(u)} \;-\; z\,\frac{\lambda}{1-e^{-\lambda}}\,\varphi_{s,z}(u)\,e^{-sD(u)}$$

$$+\,\lambda\,z\int_0^u \varphi_{s,z}(v)\,e^{-sD(v)}\,\frac{\lambda\,e^{-\lambda-\lambda v+\lambda u}}{1-e^{-\lambda}}\,dv$$

$$+\,\lambda\,z\int_u^1 \varphi_{s,z}(v)\,e^{-sD(v)}\,\frac{\lambda\,e^{-\lambda v+\lambda u}}{1-e^{-\lambda}}\,dv$$

$$=\; h'(u) \;-\; \lambda\,z\,\varphi_{s,z}(u)\,e^{-sD(u)} \;+\; \lambda\,z\int_0^1 \varphi_{s,z}(v)\,e^{-sD(v)}\,\frac{\lambda\,e^{-\lambda\langle v-u\rangle}}{1-e^{-\lambda}}\,dv.$$

Replacing the integral term above, with the aid of (6.3), yields the differential equation in $\varphi_{s,z}(u)$

$$\varphi'_{s,z}(u) \;-\; \lambda\left[1 - z\,e^{-sD(u)}\right]\varphi_{s,z}(u) \;=\; h'(u) \;-\; \lambda\,h(u), \quad u \in [0,1]. \tag{6.4}$$

The right-hand side, $\eta(u) = h'(u) - \lambda\,h(u)$, of (6.4) is readily computed for each of the integral equations (4.3), (4.4) and (4.5), using $\tilde{f}_1^+(s|D(u))$ from (6.2). Specifically, for (4.3),

$$h_G(u) \;=\; \frac{1}{s} \;-\; \frac{1}{s}\,\tilde{f}_1^+(s|D(u)) \;\;\Longrightarrow\;\; \eta_G(u) \;=\; -\frac{\lambda}{s} \;+\; \frac{\lambda}{s}\,e^{-sD(u)};$$

for (4.4),

$$h_M(u) \;=\; \frac{1}{s}\,\tilde{f}_1^+(s|D(u)) \;\;\Longrightarrow\;\; \eta_M(u) \;=\; -\frac{\lambda}{s}\,e^{-sD(u)};$$

and for (4.5),

$$h_L(u) \;=\; \tilde{f}_1^+(s|D(u)) \;\;\Longrightarrow\;\; \eta_L(u) \;=\; -\lambda\,e^{-sD(u)}.$$

To simplify the exposition, we assume a distortion-free TES$^+$ process ($D(u) \equiv u$). To solve for $\tilde{L}_X^+(z,s|u)$ in (4.5), the differential equation (6.4) now simplifies to

$$\varphi'_{s,z}(u) \;-\; \lambda\,[1 - z\,e^{-su}]\,\varphi_{s,z}(u) \;=\; -\lambda\,e^{-su}, \quad u \in [0,1], \tag{6.5}$$

with the complementary solution (see Ritger and Rose [13])

$$\varphi^{(c)}_{s,z}(u) \;=\; c\,e^{\lambda u + (\lambda\,z/s)\,e^{-su}}.$$

To determine a particular solution, define the operator

$$S_z[\psi(u)] \;=\; \psi'(u) \;-\; \lambda\left[1 - z\,e^{-\lambda u}\right]\psi(u), \quad \psi \in L^2([0,1)),$$

and postulate a solution $\varphi_{s,z}(u)$ of the form

$$\varphi_{s,z}(u) \;=\; \sum_{n=1}^{\infty} a_n\,e^{-nsu},$$

for some coefficients $a_n = a_n(s,z)$. Applying the operator S_z to the postulated solution above then results in

$$S_z[\varphi_{s,z}(u)] \;=\; -(\lambda+s)\,a_1\,e^{-su} \;+\; \sum_{n=2}^{\infty}[-(\lambda+ns)\,a_n + \lambda\,z\,a_{n-1}]\,e^{-nsu}.$$

But from (6.3), $S_z[\varphi_{s,z}(u)] = -\lambda\,e^{-su}$, implying the recursive relations $a_1 = \dfrac{\lambda}{\lambda+s}$ and $a_n = \dfrac{\lambda\,z}{\lambda+n\,s}\,a_{n-1}$, for $n \geq 2$. Hence, $a_n = \dfrac{1}{z}\displaystyle\prod_{k=1}^{n}\dfrac{\lambda\,z}{\lambda+k\,s}$, and the solution of (6.5) can now be written explicitly in terms of the constant c as

$$\tilde{L}(z,s|u) \;=\; c\,e^{\lambda u + (\lambda z/s)\,e^{-su}} \;+\; \frac{1}{z}\sum_{n=1}^{\infty} e^{-nsu}\prod_{k=1}^{n}\frac{\lambda\,z}{\lambda+k\,s}. \tag{6.6}$$

The constant $c = c(z,s,\lambda)$ is independent of u, and will be determined by substituting (6.6) into (6.3). A direct calculation of $h_L(u) = \tilde{f}_1^{+}(s|u)$, with the aid of (6.2), yields

$$h_L(u) \;=\; \frac{\lambda}{s+\lambda}\left[e^{-su} + \frac{1-e^{-s}}{1-e^{-\lambda}}\,e^{\lambda(u-1)}\right]. \tag{6.7}$$

Substituting (6.6) and (6.7) into (6.3) results in

$$\begin{aligned}
c\,e^{\lambda u + (\lambda z/s)\,e^{-su}} &+ \frac{1}{z}\sum_{n=1}^{\infty} e^{-nsu}\prod_{k=1}^{n}\frac{\lambda\,z}{\lambda+k\,s} \;=\\
&= \frac{\lambda}{\lambda+s}\left[e^{-su} + \frac{1-e^{-s}}{1-e^{-\lambda}}\,e^{\lambda(u-1)}\right]\\
&+ \frac{c\,z\,\lambda}{1-e^{-\lambda}}\int_0^1 e^{\lambda v + (\lambda z/s)e^{-sv} - sv - \lambda\langle v-u\rangle}\,dv\\
&+ \frac{\lambda}{1-e^{-\lambda}}\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n}\frac{\lambda\,z}{\lambda+k\,s}\right]\int_0^1 e^{-nsv - sv - \lambda\langle v-u\rangle}\,dv.
\end{aligned}$$

Since c does not depend on u, set in particular, $u=0$ in the equation above, yielding

$$\begin{aligned}
c\,e^{(\lambda z/s)} &+ \frac{1}{z}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\frac{\lambda\,z}{\lambda+k\,s} \;=\\
&= \frac{\lambda}{\lambda+s}\,\frac{e^{\lambda}-e^{-s}}{e^{\lambda}-1}\\
&+ \frac{c\,z\,\lambda}{1-e^{-\lambda}}\int_0^1 e^{(\lambda z/s)\,e^{-sv} - sv}\,dv\\
&+ \frac{\lambda}{1-e^{-\lambda}}\sum_{n=1}^{\infty}\frac{1-e^{-\lambda-(n+1)s}}{\lambda+(n+1)s}\prod_{k=1}^{n}\frac{\lambda\,z}{\lambda+k\,s}.
\end{aligned}$$

Using the evaluation

$$\int_0^1 e^{(\lambda z/s)\, e^{-sv} - sv}\, dv \;=\; \frac{e^{(\lambda z/s)} - e^{(\lambda z/s)\, e^{-s}}}{\lambda z}$$

in the preceding equation, we conclude

$$c \;=\; \frac{\lambda \dfrac{e^{\lambda} - e^{-s}}{\lambda + s} + \displaystyle\sum_{n=1}^{\infty} \left(\lambda \frac{e^{\lambda} - e^{-(n+1)s}}{\lambda + (n+1)s} - \frac{e^{\lambda} - 1}{z} \right) \prod_{k=1}^{n} \frac{\lambda\, z}{\lambda + k\, s}}{e^{(\lambda z/s) e^{-s} + \lambda} - e^{(\lambda z/s)}}. \tag{6.8}$$

The generating function $\tilde{L}_X^+(z, s|u)$ is now given by (6.6) with c given in (6.8).

The solutions for $\tilde{M}_X^+(s|u)$ and $\tilde{G}_X^+(z, s|u)$ are largely similar. However, for $\tilde{M}_X^+(s|u)$, we have a straightforward solution in terms of $\tilde{L}_X^+(z, s|u)$, as given by (3.7).

7 THE PEAKEDNESS OF TES$^+$ TRAFFIC PROCESSES

The peakedness functional provides a partial characterization of the burstiness of an ergodic traffic stream by gauging its effect when offered to an infinite server group. In practice, peakedness is typically used to approximate a solution for blocking and delay statistics in finite-server queues.

Let $\{X_n\}$ be a stationary sequence of interarrival times with a general probability law, arrival rate $\lambda = 1/E[X_n] < \infty$, and expectation function $M(t)$ (recall Section 1). Assume that the corresponding traffic stream is offered to an infinite server group consisting of independent servers with common service time distribution F. Let $B(t)$ be the number of busy servers at time t, and assume that its limiting statistics exist. The peakedness functional, z_X, associated with the traffic process $\{X_n\}$, given by

$$z_X[F] \;=\; \lim_{t \uparrow \infty} \frac{Var[B(t)]}{E^2[B(t)]}, \tag{7.1}$$

maps the space of all service time distributions to non-negative numbers.

Let $\{F_\mu\}$ be a parametric family of service time distributions, indexed by the service rate $\mu = 1/\int_0^\infty x\, dF_\mu(x)$. It is convenient to standardize the $F_\mu(x)$ to unit rate by defining $F_1(x) = F_\mu(x/\mu)$, and to replace the peakedness functional $z_X[F_\mu]$ from (7.1) by the corresponding peakedness function

$$z_{X,F_1}(\mu) \;=\; z_X[F_\mu]. \tag{7.2}$$

Interestingly, if $\{G_\mu\}$ is any other parametric family of service time distributions, then the corresponding peakedness functions $z_{X,F_1}(\mu)$ and $z_{X,G_1}(\mu)$ contain equivalent information on the traffic process $\{X_n\}$, in the sense that the two are connected by a known transformation (see Jagerman [7]). In particular, for an exponential service time distribution, $F_1(x) = 1 - e^{-x}$, the corresponding peakedness function (7.2) is denoted by $z_{X,exp}(\mu)$, and has the representation (see Eckberg [3]),

$$z_{X,exp}(\mu) \;=\; 1 - \frac{\lambda}{\mu} + \mu\, \tilde{M}(\mu). \tag{7.3}$$

Consider the auxiliary peakedness function

$$z_{X,exp}(\mu, x) \;=\; 1 - \frac{\lambda}{\mu} \;+\; \mu\, \tilde{M}(\mu|x), \tag{7.4}$$

from which (7.3) can be obtained by integration with respect to the interarrival time density $f_X(x)$. Substituting the integral equation (3.4) into (7.4) yields the integral equation

$$z_{X,exp}(\mu, x) \;=\; 1 \;-\; \frac{\lambda}{\mu} \;+\; \frac{\lambda}{\mu}\, \tilde{f}_1(\mu|x) \;+\; \int_0^\infty z_{X,exp}(\mu, x)\, e^{-\mu y}\, f_1(y|x)\, dy. \tag{7.5}$$

For the remainder of this section, we specialize the discussion to TES$^+$ processes $\{X_n^+\}$. In this case, the peakedness value, $z_{exp}^+(0)$, assumes a particularly simple form. However, before stating the main result, we shall need the following simple facts which will serve to simplify the proof.

Proposition 1 *For any TES$^+$ process $\{X_n^+\}$ of the form (1.3),*

$$z_{exp}^+(\mu) \;=\; \int_0^1 z_{exp}^+(\mu, D(v))\, dv, \tag{7.6}$$

$$\frac{d}{d\mu} z_{exp}^+(\mu) \;=\; \int_0^1 \frac{\partial}{\partial \mu} z_{exp}^+(\mu, D(v))\, dv, \tag{7.7}$$

$$\tilde{f}_1^+(\mu|D(u)) \;=\; 1 \;-\; \mu \int_0^1 D(v)\, g_U^+(v|u)\, dv \;+\; o(\mu), \tag{7.8}$$

$$\tilde{f}_X^+(\mu) \;=\; 1 \;-\; \frac{\mu}{\lambda} \;+\; \frac{1}{2} m_2^+\, \mu^2 \;+\; o(\mu^2), \tag{7.9}$$

where $m_2^+ = E[(X_n^+)^2]$.

Proof (7.6) follows immediately from (7.4), because the marginal density of background TES processes is uniform on $[0, 1)$.

To prove (7.7) we show that $z_{exp}^+(\mu, x)$ is analytic at $\mu = 0$, permitting the interchange of integration and differentiation. To this end, use the asymptotic expansion of $\tilde{M}_X^+$ from (4.6) in the general relation (7.4) to deduce

$$z_{exp}^+(\mu, D(u)) \;\sim\; 1 \;+\; b_X^+(u) \;+\; O(\mu), \quad \mu \to 0,$$

proving that, in fact, $z_{exp}^+(\mu, x)$ is analytic for $Re[\mu] \geq 0$. Moreover, at $\mu = 0$,

$$b_X^+(u) \;=\; z_{exp}^+(0, D(u)) \;-\; 1,$$

and from (7.6),

$$b_0^+ \;=\; \int_0^1 b_X^+(u)\, du \;=\; z_{exp}^+(0) \;-\; 1.$$

This determines the constant b_0^+ (which was left undetermined in Theorem 2) in terms of $z_{exp}^+(0)$. The determination of $z_{exp}^+(0)$ will be given later in Theorem 3.

Equation (7.8) is obtained by expanding $\tilde{f}_1^+(\mu|D(u)) = \int_0^1 e^{-\mu D(v)}\, g_U^+(v|u)\, dv$ in powers of μ around 0. Similarly, (7.9) is obtained by expanding $\tilde{f}_X^+(\mu) = E[e^{\mu X_n^+}]$ in powers of μ around 0. □

The main result now follows.

Theorem 3 *For any TES$^+$ process $\{X_n^+\}$ of the form (1.3), we have the representations*

$$z_{exp}^+(0) \;=\; \frac{1+(c_X^+)^2}{2} \;+\; (c_X^+)^2 \sum_{\tau=1}^{\infty} \rho_X^+(\tau) \;=\; \frac{1+(c_X^+)^2\, \pi\, s_X^+(0)}{2}, \tag{7.10}$$

where $(c_X^+)^2 = Var[X_n^+]/E^2[X_n^+]$ is the squared coefficient of variation corresponding to f_X^+.

Proof Substituting $x = D(u)$, $y = D(v)$ and (4.2) into (7.5) yields

$$z_{exp}^+(\mu, D(u)) \;=\; 1 - \frac{\lambda_X^+}{\mu} + \frac{\lambda_X^+}{\mu}\, \tilde{f}_1^+(\mu | D(u)) \;+\; \int_0^1 z_{exp}^+(\mu, D(v))\, e^{-\mu D(v)}\, g_U^+(v|u)\, dv. \tag{7.11}$$

To obtain $z_{exp}^+(0, D(u))$, substitute (7.8) into (7.11) and set $\mu = 0$. This yields the integral equation

$$z_{exp}^+(0, D(u)) \;=\; 1 \;-\; \lambda_X^+ \int_0^1 D(v)\, g_U^+(v|u)\, dv \;+\; \int_0^1 z_{exp}^+(0, D(v))\, g_U^+(v|u)\, dv, \tag{7.12}$$

in the unknown function $z_{exp}^+(0, D(u))$. Since $g_U^+(v|u) = g_U(v-u)$ is periodic in each argument, we make use of Fourier series to represent the solution of (7.12). Substituting the representation (4.1) for $g_U^+(v|u)$ into the integral equation (7.12) and interchanging summation and integration, we obtain after some manipulation

$$\begin{aligned} z_{exp}^+(0, D(u)) \;=\; 1 - &\sum_{\nu=-\infty}^{\infty} \lambda_X^+\, \tilde{f}_V(i2\pi\nu)\, \tilde{D}(-i2\pi\nu)\, e^{-i2\pi\nu u} \\ &+ \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)\, e^{-i2\pi\nu u} \int_0^1 e^{i2\pi\nu v}\, z_{exp}^+(0, D(v))\, dv. \end{aligned} \tag{7.13}$$

Next, we let $\hat{z}_\nu = \int_0^1 e^{-i2\pi\nu v}\, z_{exp}^+(0, D(v))\, dv$, and conclude that $z_{exp}^+(0, D(v))$ has the Fourier series representation

$$z_{exp}^+(0, D(u)) \;=\; \sum_{\nu=-\infty}^{\infty} \hat{z}_\nu\, e^{i2\pi\nu u}. \tag{7.14}$$

The representation (7.14) is now used on both sides of (7.13) to obtain, after conjugation,

$$\begin{aligned} \sum_{\nu=-\infty}^{\infty} \hat{z}_\nu\, e^{i2\pi\nu u} \;=\; 1 - &\sum_{\nu=-\infty}^{\infty} \lambda_X^+\, \tilde{f}_V(-i2\pi\nu)\, \tilde{D}(i2\pi\nu)\, e^{i2\pi\nu u} \\ &+ \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\, \hat{z}_\nu\, e^{i2\pi\nu u}. \end{aligned} \tag{7.15}$$

The Fourier coefficients, $\hat{z}_\nu$, can now be deduced to be

$$\hat{z}_\nu \;=\; \begin{cases} z_{exp}^+(0), & \nu = 0 \\[2ex] -\dfrac{\lambda_X^+\, f_V(-i2\pi\nu)}{1 - \tilde{f}_V(-i2\pi\nu)}\, \tilde{D}(i2\pi\nu), & \nu \neq 1 \end{cases} \tag{7.16}$$

where the first case above follows by setting (7.14) in (7.6) for $\nu = 0$, and the second case above results from equating Fourier coefficients in (7.15) for $\nu \neq 0$. Furthermore, equating coefficients there for $\nu = 0$ yields the relation $\hat{z}_0 = 1 - \lambda_X^+ \tilde{D}(0) + \hat{z}_0$, which is consistent with the fact that $\tilde{D}(0) = 1/\lambda_X^+$, but $\hat{z}_0 = z_{exp}^+(0)$ cannot yet be deduced. Substituting, however, the Fourier coefficients, $\hat{z}_\nu$, from (7.16) into the right-hand side of (7.14) now yields the relation

$$z_{exp}^+(0, D(u)) \;=\; z_{exp}^+(0) \;-\; \lambda_X^+ \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \frac{\tilde{f}_V(-i2\pi\nu)}{1 - \tilde{f}_V(-i2\pi\nu)} \, \tilde{D}(i2\pi\nu)\, e^{i2\pi\nu u}, \tag{7.17}$$

with $z_{exp}^+(0)$ as yet undetermined.

To determine $z_{exp}^+(0)$, integrate (7.11) with respect to u, and use the relation (7.6) to obtain

$$z_{exp}^+(\mu) \;=\; 1 \;-\; \frac{\lambda_X^+}{\mu} \left[1 - \tilde{f}_1^+(\mu)\right] \;+\; \int_0^1 z_{exp}^+(\mu, D(v))\, e^{-\mu D(v)} dv. \tag{7.18}$$

Next, substitute (7.9) into the right-hand side of (7.18) and expand the resulting equation in powers of μ around 0, which yields

$$\begin{aligned} & z_{exp}^+(0) \;+\; \mu \frac{d}{d\mu} z_{exp}^+(0) \;+\; o(\mu) \;= \\ & \quad \frac{1}{2} \lambda_X^+ \, \mu \, m_2^+ \;+\; \int_0^1 \left[z_{exp}^+(0, D(v)) + \mu \frac{\partial}{\partial\mu} z_{exp}^+(0, D(v)) \right] \left[1 - \mu\, D(v) \right] dv \;+\; o(\mu). \end{aligned}$$

Simplifying the above equation with the aid of (7.6) and (7.7) and dividing by μ now gives the following condition equation for $z_{exp}^+(0, D(u))$,

$$\int_0^1 z_{exp}^+(0, D(u))\, D(u)\, du \;=\; \frac{1}{2} \lambda_X^+ \, m_2^+. \tag{7.19}$$

The value of $z_{exp}^+(0)$ is finally obtained, by substituting (7.17) into (7.19), resulting in

$$\begin{aligned} z_{exp}^+(0) \;&=\; \frac{1}{2} (\lambda_X^+)^2 \, m_2^+ \;+\; (\lambda_X^+)^2 \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \frac{\tilde{f}_V(-i2\pi\nu)}{1 - \tilde{f}_V(-i2\pi\nu)} \, |\tilde{D}(i2\pi\nu)|^2 \\ &=\; \frac{1}{2} (\lambda_X^+)^2 \, m_2^+ \;+\; (\lambda_X^+)^2 \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1 - \tilde{f}_V(i2\pi\nu)} \, |\tilde{D}(i2\pi\nu)|^2, \end{aligned} \tag{7.20}$$

where the second equality is justified by the fact that all qauntities in (7.20) are real except for the terms in the infinite sums.

The first equality of (7.10) follows from (7.20), noting that

$$\begin{aligned} \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1 - \tilde{f}_V(i2\pi\nu)} \, |\tilde{D}(i2\pi\nu)|^2 \;&=\; \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \sum_{\tau=1}^{\infty} \tilde{f}_V^{\tau}(i2\pi\nu) \, |\tilde{D}(i2\pi\nu)|^2 \\ &=\; \sum_{\tau=1}^{\infty} \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \tilde{f}_V^{\tau}(i2\pi\nu) \, |\tilde{D}(i2\pi\nu)|^2 \;=\; Var[X_n^+] \sum_{\tau=1}^{\infty} \rho_X^+(\tau), \end{aligned}$$

where the second equality is justified by the absolute convergence of the series, and the third by appeal to (1.6). Finally, the second equality of (7.10) follows from the identity

$$\sum_{\tau=1}^{\infty} \rho_X^+(\tau) \;=\; \frac{\pi s_X^+(0) - 1}{2},$$

in view of (1.2). □

Theorem 3 reveals an interesting connection between the peakedness $z_{exp}^+(0)$ and the index of dispersion notion of traffic burstiness or variability [2, 4, 5]. Let $\{X_n\}$ be a stationary sequence of interarrival times. The index of dispersion for intervals (IDI) is the sequence $\{I_n\}$, defined by

$$I_n \;=\; \frac{Var(X_{j+1} + \ldots + X_{j+n})}{n\,E[X_j]} \;=\; c_X^2 \left[1 + 2 \sum_{\tau=1}^{n-1} \left(1 - \frac{\tau}{n} \right) \rho_X(\tau) \right]. \tag{7.21}$$

The limit $I_X = \lim_{n \uparrow \infty} I_n$ of (7.21) is

$$I_X \;=\; c_X^2 \left[1 + 2 \sum_{\tau=1}^{\infty} \rho_X(\tau) \right]. \tag{7.22}$$

For a TES$^+$ process, $\{X_n^+\}$, a comparison of (7.22) with (7.10) yields the relation

$$I_X^+ \;=\; 2\, z_{exp}^+(0) \;-\; 1. \tag{7.23}$$

Thus, for TES$^+$ processes, $z_{exp}^+(0)$ and I_X^+ convey the same information. However, for a general traffic process, $\{X_n\}$, the peakedness function, $z_{X,exp}(\mu)$, contains much more information,and especially the traffic impact on a server system. It is, therefore, desirable to obtain at least an approximate solution of the integral equation (7.11), for $\mu > 0$.

To this end, define

$$d_\nu(\mu) \;=\; \int_0^1 e^{-i2\pi\nu v - \mu D(v)}\, dv, \tag{7.24}$$

and for the auxiliary peakedness function, $z_{exp}^+(\mu, D(u))$ in (7.4), assume an approximation of the form

$$z_{exp}^+(\mu, D(u)) \;\simeq\; z_{exp}^+(\mu) \;+\; \frac{\lambda_X^+}{\mu} \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} b_\nu\, d_\nu(\mu)\, e^{i2\pi\nu u}, \tag{7.25}$$

for some coefficients b_ν, as yet undetermined. (7.25) clearly satisfies (7.6), independently of the choice of the b_ν. Next, we make the essential assumption that the coefficients b_ν depend only on ν, and require that the approximation (7.25) satisfy (7.18). Accordingly, on substituting the former into the latter, we get

$$z_{exp}^+(\mu) \;\simeq\; \frac{1}{1 - \tilde{f}_X^+(\mu)} \;-\; \frac{\lambda_X^+}{\mu} \;+\; \frac{\lambda_X^+}{\mu}\, \frac{1}{1 - \tilde{f}_X^+(\mu)} \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} b_\nu\, |d_\nu(\mu)|^2. \tag{7.26}$$

In order to determine the b_ν, we further require that sending $\mu \downarrow 0+$ in (7.26), result in a limit that exactly equals the correct value, $z^+_{exp}(0)$, as given in Theorem 3. We show that this requirement yields

$$\lim_{\mu \downarrow 0+} z^+_{exp}(\mu) \;=\; \frac{1}{2}\,(\lambda^+_X)^2\, m^+_2 \;+\; (\lambda^+_X)^2 \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} b_\nu \, |\tilde{D}(i2\pi\nu)|^2. \tag{7.27}$$

To see that, use (7.9) to obtain the evaluation $\dfrac{1}{1-\tilde{f}^+_X(\mu)} \;-\; \dfrac{\lambda^+_X}{\mu} \;=\; \dfrac{1}{2}\,(\lambda^+_X)^2\, m^+_2 \;+\; o(\mu)$, and expand (7.24) in powers of μ about 0 to deduce the relation $d_\nu \;=\; -\mu\, \tilde{D}(i2\pi\nu) + o(\mu)$. Equation (7.27) follows from the preceding two relations. A comparison of (7.27) with (7.20) now determines the b_ν as

$$b_\nu \;=\; \frac{\tilde{f}_V(i2\pi\nu)}{1-\tilde{f}_V(i2\pi\nu)}.$$

Substituting the b_ν above into (7.25) yields the final approximation,

$$z^+_{exp}(\mu, D(u)) \;\simeq\; z^+_{exp}(\mu) \;+\; \frac{\lambda^+_X}{\mu} \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1-\tilde{f}_V(i2\pi\nu)}\, d_\nu(\mu)\, e^{iz\pi\nu u}, \tag{7.28}$$

and similarly, from (7.26),

$$z^+_{exp}(\mu) \;\simeq\; \frac{1}{1-\tilde{f}^+_X(\mu)} \;-\; \frac{\lambda^+_X}{\mu} \;+\; \frac{\lambda^+_X}{\mu}\,\frac{1}{1-\tilde{f}^+_X(\mu)} \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1-\tilde{f}_V(i2\pi\nu)}\, |d_\nu(\mu)|^2. \tag{7.29}$$

It is interesting to observe that the first two terms,

$$\frac{1}{1-\tilde{f}^+_X(\mu)} \;-\; \frac{\lambda^+_X}{\mu},$$

in (7.29) constitute the exact peakedness function for the corresponding renewal traffic (i.e., the iid interarrival time process with marginal density f^+_X). Thus, these first two terms represent the contribution of the marginal density, f^+_X, to the burstiness of $\{X^+_n\}$, while the third term in (7.29) arises from the dependence structure of $\{X^+_n\}$. Equation (7.29) thus captures the effect of both first-order and higher-order statistics on the burstiness of a TES$^+$ traffic process. This interpretation is consistent with the fact that a renewal process entirely lacks second-order effects (its autocorrelation function is zero for positive lags), and consequently, renewal traffic owes its peakedness to first-order statistics, exclusively. Interestingly, the case $\mu = 0$ in Equation (7.10) exhibits precisely two contributions, namely, those corresponding to first-order and second-order statistics, but no higher ones.

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