THE TRANSITION AND AUTOCORRELATION STRUCTURE OF TES PROCESSES

Part I: General Theory

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ABSTRACT

TES (Transform-Expand-Sample) is a versatile class of stochastic sequences which can capture arbitrary marginals and a wide variety of sample path behavior and autocorrelation functions. In TES, the initial variate is uniform on [0,1) and the next variate is obtained recursively by taking the fractional part (i.e., modulo-1 reduction) of a linear autoregressive scheme. We show how this class gives rise to uniform Markovian sequences in a general and natural way, by observing that marginal uniformity is closed under modulo-1 addition of an independent variate with arbitrary distribution. We derive the transition function of TES sequences and the autocovariance function of transformed TES sequences using Fourier and Laplace Transform methods. The autocovariance formulas are amenable to fast and accurate calculation and provide the theoretical basis for a computer-based methodology of heuristic TES modeling of empirical data. A companion paper contains various examples which show the efficacy of the TES approach by comparing numerical and simulation-based calculations for a variety of TES autocorrelation functions. The results have applications to the modeling of autocorrelated sequences, particularly in a Monte Carlo simulation context.

Keywords and Phrases: autocorrelated variates, correlated variates, autocorrelation function, autocovariance function, Markov processes, autoregressive processes, Monte Carlo simulation, TES Processes and Methods

1 INTRODUCTION

This paper is motivated mainly by the need to model and generate autocorrelated sequences of variates as input to Monte Carlo simulations. Furthermore, when an empirical time series (sample) is available, one would like the model to simultaneously capture the empirical sample's marginal distribution and autocorrelation function. See, for example, Liu and Munson (1982) and references cited therein.

A prime application area is queueing systems where the standard modeling assumptions usually require general independence (GI) of inter-arrival intervals as well as service times; the term "general" means that the respective marginal distributions are not restricted to a particular form. A natural way to generalize the GI assumption is to introduce dependencies among inter-arrival intervals and service times. Intuition suggests that positively autocorrelated inter-arrival intervals and service times would increase the burstiness of the arrival and service processes, and "worsen" such performance measures as mean number in the system and average waiting time. A growing body of literature has indeed shown that autocorrelation can have a profound effect on system performance. See, for example, Jacobs (1978) and (1980), Latouche (1985), Tin (1985), Heffes and Lucantoni (1986), Patuwo and Disney (1989), and references cited therein. Autocorrelated traffic has been notably addressed in Sriram and Whitt (1986) and Fendick et al. (1989) in the context of packet traffic in high-speed communications networks.

A broad paradigm for generating autocorrelated stochastic sequences is comprised of *foreground/background* schemes (see Figure 1). Such schemes generate two autocorrelated sequences in lockstep: an auxiliary sequence $\{Y_n\}$ (the *background* sequence) and the target sequence $\{X_n\}$ (the *foreground* sequence). The background sequence $\{Y_n\}$ is usually a stationary process generated recursively by some transition function $Y_{n+1} = T_G(Y_n, Z_n)$, where $\{Z_n\}$ is an i.i.d. *(independent identically distributed)* Uniform(0,1) sequence (the Z_n form a pseudo-random number stream, available on most computers). The transition function T_G is chosen to preserve the marginal distribution G of the initial Y_0, and to render $\{Y_n\}$ c.i.d. *(correlated identically distributed)*. The sequence $\{X_n\}$ is obtained in lockstep from the sequence $\{Y_n\}$ via $X_n = D(Y_n)$, where the mapping D is called a *distortion* for historical reasons (Grenander and Rosenblatt (1957), Chapter 1). If

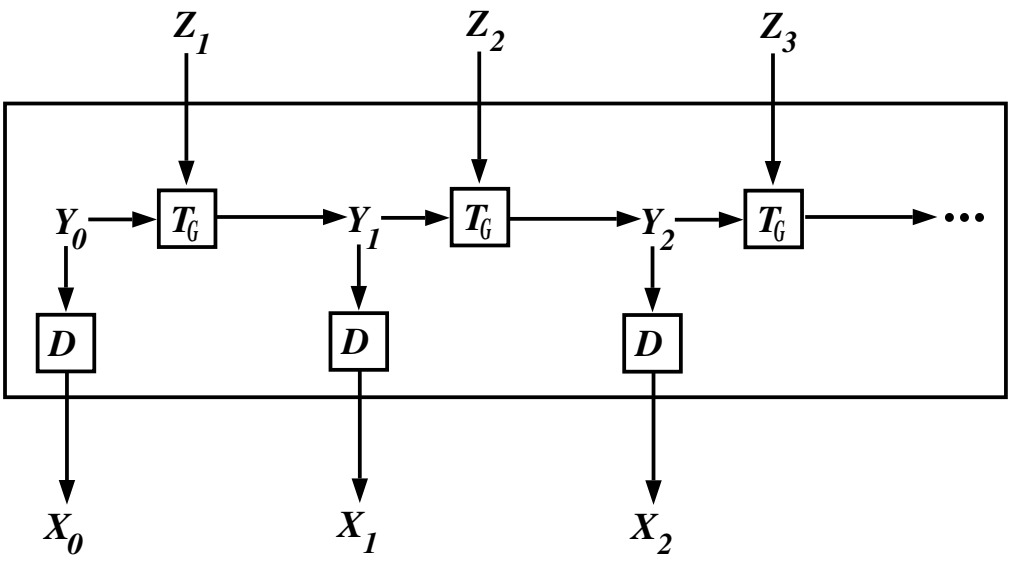

Figure 1: The Foreground/Background Paradigm

D is the identity ($\{Y_n\}$ and $\{X_n\}$ coincide), then the scheme is simple; otherwise, it is compound. A typical compound scheme employs $D = F^{-1} \circ G$, where G is the marginal distribution of Y_n, F is that of X_n and the circle denotes functional composition. Here D is a two stage mapping: the first stage G operates on the background variate Y_n and yields a Uniform$(0, 1)$ variate $U_n = G(Y_n)$; the second stage executes the inversion method (Brateley et al. (1987), Devroye (1986)) to yield the foreground variate $X_n = F^{-1}(U_n)$ with the requisite marginal distribution F. Non-stationary sequences fall within the foreground/background paradigm in a natural way by introducing time dependent transition functions and distortions.

The literature contains considerable work on generating target sequences $\{X_n\}$ with simple or compound schemes. Especially noteworthy is the work of P.A.W. Lewis, B. Liu, B. Schmeiser and their co-workers. Some representative references of simple schemes include Lawrance and Lewis (1981), Dewald and Lewis (1985), and Lewis, McKenzie and Hugus (1988) which treat, respectively, exponential, Laplace and gamma marginals. See also Melamed (1991) for more details. Such methods are tailored to a specific target distribution and have easily derived auto-correlation functions, but require one to devise a specific transition function T_F for each marginal distribution F. Compound schemes are more general, but they can usher in mathematical and computational complexities due to the complexity of the

distortion D; see Liu and Munson (1982) and Schmeiser (1990). Since compound schemes generate intermediate uniform variates, there is a distinct advantage in dealing with a uniform background sequence; the distortion reduces then to the inverse F^{-1}. The problem of generating autocorrelated uniform sequences $\{U_n\}$ was addressed by Lewis and McKenzie (1991) which defines the minification (moving minimum) and maxification (moving maximum) processes.

The subject of this paper is a class of foreground/uniform background schemes, called TES, first introduced in Melamed (1991) and studied mainly by simulation. TES processes were shown there to have low generation complexity and versatile sample path behavior – their autocorrelation functions admitting a variety of functional forms including monotone, oscillatory, alternating and combinations thereof. The acronym TES (*Transform-Expand-Sample*) summarizes the geometric construction of a TES process on the unit circle (unit circumference), where the next iterate is obtained from the previous one in three steps: first, the previous iterate is mapped to a point on the unit circle (*Transform*); second, the point is mapped to an interval (*Expand*); and third, a value is sampled uniformly in that interval independently of the probabilistic past (*Sample*). The geometric interpretation of the Transform and Expand steps corresponds to modulo-1 arithmetic, which is employed to preserve the uniformity of the basic TES process (see Section 3).

In this paper we introduce considerable generality to the definitions in Melamed (1991). First we generalize the recursive construction of the underlying TES processes; and second we permit arbitrary distortions. The main contribution of this paper is the derivation of numerically computable formulas for TES processes and their distorted variants. The formulas allow very fast and accurate calculation of TES autocorrelations, thereby providing the theoretical basis for heuristic fitting of TES models to empirical data. A visual interactive software environment for TES modeling is described in Geist and Melamed (1991) and (1992).

The remainder of the paper is organized as follows. Section 2 establishes notation and presents some preliminary material. Section 3 defines and explains TES processes. Section 4 derives the (Markovian) transition function of TES processes, and Section 5 derives numerically computable formulas for TES autocovariance and autocorrelation functions. A brief conclusion in Section 6 terminates the theoretical

portion of this work. A companion paper, Jagerman and Melamed (1992) (Part II), works out various specializations of the general formulas derived in Section 5 and demonstrates the efficacy of the numerically computable formulas by comparing them to simulation-based statistics.

2 PRELIMINARIES

Throughout this paper we adopt the following notation. The indicator function of set A is written $1_A(x)$, or $1[x \in A]$ when A is notationally cumbersome. The Laplace Transform of a function $f(x)$ is consistently denoted by appending a tilde to the function symbol, i.e., $\tilde{f}(s) = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$. The modulo-1 operator is denoted $\langle\cdot\rangle$ and defined for any real x by $\langle x\rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor = \max\{integer\ n : n \leq x\}$. The circular (modulo-1) interval $C(x, a, b)$ with offsets a and b about x is defined for any $x, a, b \in [0, 1)$ by

$$C(x,a,b) = \begin{cases} [\langle x+a\rangle, \langle x+b\rangle), & if\ \ \langle x+a\rangle < \langle x+b\rangle \\ [0,1) \setminus [\langle x+b\rangle, \langle x+a\rangle), & if\ \ a \neq b,\ \ \langle x+a\rangle \geq \langle x+b\rangle \\ \{\langle x+a\rangle\}, & if\ \ a = b \end{cases} \quad (2.1)$$

The autocovariance and autocorrelation functions of a stationary process $\{X_n\}_{n=0}^{\infty}$ with finite positive second moment are defined, respectively, for lags $\tau = 0, 1, 2, \ldots$ by

$$\psi_X(\tau) \ = \ E[X_0 X_\tau] - \mu_X^2, \ \ \rho_X(\tau) \ = \ \frac{\psi_X(\tau)}{\sigma_X^2}, \quad (2.2)$$

where μ_X and σ_X^2 are the common mean and variance, respectively, of the X_n.

The basis for our approach is provided by a simple fact (Lemma 1 below), establishing that uniformity is closed under modulo-1 addition. While the result (dubbed here *General Iterated Uniformity*) is not new (see, e.g., Feller (1971), p. 64; Jagerman (1963) for a deterministic version; L'Ecuyer (1988) for a discrete state variant), we feel that its importance merits the inclusion of its proof for completeness.

Lemma 1 (General Iterated Uniformity) *Let $U \sim$ Uniform$(0, 1)$ and let further $X = \langle U + V\rangle$, where V is an arbitrarily distributed random variable independent of U. Then $X \sim$ Uniform$(0, 1)$.*

Proof We first show that $X = \langle U+c \rangle \sim$ Uniform$(0,1)$ for any real c. It is enough to assume $c \in [0,1)$, because $\langle U+c \rangle = \langle U+ \langle c \rangle \rangle$, where $\langle c \rangle \in [0,1)$. For $x \in [0,1]$,

$$P\{X \le x\} = P\{U+c < 1,\ U+c \le x\} + P\{U+c \ge 1,\ U+c-1 \le x\} = A + B,$$

where,

$$\begin{aligned} A &= P\{U < 1-c,\ U \le x-c\} = (x-c)\, 1_{[c,1)}(x) \\ B &= P\{1-c \le U \le x-c+1\} = x\, 1_{[0,c)}(x) + c\, 1_{[c,1)}(x) \end{aligned}$$

Summing the above, we get

$$P\{X \le x\} = A + B = x$$

proving that $X \sim$ Uniform(0,1). Next, for arbitrary V and $x, y \in [0,1]$,

$$P\{X \le x \mid V = y\} = P\{\langle U + y \rangle \le x\} = x$$

is independent of y. Hence, $P\{X \le x\} = x$ for all $x \in [0,1]$, which is the claimed result. □

A basic result we shall employ to study autocorrelation properties is the Poisson summation formula (Lighthill (1960)) which states that if g is a function of bounded variation such that $\int_{-\infty}^{\infty} g(x)dx < \infty$, then

$$\sum_{n=-\infty}^{\infty} g(x+n) = \sum_{\nu=-\infty}^{\infty} e^{i2\pi\nu x} \int_{-\infty}^{\infty} e^{-i2\pi\nu y}\, g(y)dy = \sum_{\nu=-\infty}^{\infty} e^{i2\pi\nu x}\, \tilde{g}(i2\pi\nu) \tag{2.3}$$

$(i = \sqrt{-1})$. The righthand side of (2.3) is the Fourier series of the periodic function $h(x) = \sum_{n=-\infty}^{\infty} g(x+n)$, and the integrals are the corresponding Fourier coefficients.

Finally, the following simple lemma will be frequently used.

Lemma 2 *Let $h(x)$ be a real function whose Laplace Transform $\tilde{h}(s)$ exists. Then, $\tilde{h}(ir)$ and $\tilde{h}(-ir)$ are complex conjugates for any real r.*

Proof The complex exponential pairs $e^{ir} = \cos(r) + i\sin(r)$ and $e^{-ir} = \cos(r) - i\sin(r)$ are obviously complex conjugates. Since $\tilde{h}(ir)$ and $\tilde{h}(-ir)$ are obtained by integrating these with respect to $h(x)dx$, it follows that $\tilde{h}(ir)$ and $\tilde{h}(-ir)$ are also complex conjugates. □

3 TES PROCESSES

Motivated by Melamed (1991), we consider two related classes of TES methods (algorithmic constructions) called TES$^+$ and TES$^-$. The attendant TES processes $\{U_n^+\}_{n=0}^{\infty}$ and $\{U_n^-\}_{n=0}^{\infty}$, respectively, are defined over a common probability space, and given recursively by

$$U_n^+ = \begin{cases} U_0 \,, & n \;=\; 0 \\ \\ \langle U_{n-1}^+ \;+\; V_n \rangle \,, & n \;>\; 0 \end{cases} \tag{3.1}$$

$$U_n^- = \begin{cases} U_n^+ \,, & n \textit{ even} \\ \\ 1 \;-\; U_n^+ \,, & n \textit{ odd} \end{cases} \tag{3.2}$$

where $U_0^+ \;=\; U_0^- \;=\; U_0 \sim$ Uniform$(0,1)$, and $\{V_n\}_{n=0}^{\infty}$ is a stationary innovation sequence, i.e., a sequence of i.i.d. variates where each V_n is independent of $\{U_0^+ \,, \ldots, \, U_{n-1}^+\}$ and $\{U_0^- \,, \ldots, \, U_{n-1}^-\}$. We distinguish between TES$^+$ and TES$^-$ in order to provide coverage of the entire lag-1 autocorrelation range for marginally uniform sequences: TES$^+$ methods (appropriately parameterized) realize all values of the first autocorrelation in the range [0,1], while their TES$^-$ counterparts cover the range [-1,0] (see Melamed (1991)). In a similar vein, distorted TES sequences cover realizable positive and negative ranges of their first autocorrelation.

From now on and without further mention, we shall consistently append the proper superscript (plus or minus) to distinguish between similar mathematical objects associated with TES$^+$ and TES$^-$, respectively. For example, $\rho_U^+(\tau)$ and $\rho_U^-(\tau)$ will denote the autocorrelation coefficient functions of $\{U_n^+\}$ and $\{U_n^-\}$, respectively. The more precise equivalent notation $\rho_{U^+}(\tau)$ and $\rho_{U^-}(\tau)$ is more cumbersome and will not be used. Objects common to $\{U_n^+\}$ and $\{U_n^-\}$, or those for which the distinction is not important, will carry no superscript. For example, if $\{X_n^+\}$ and $\{X_n^-\}$ are the stationary distorted sequences given by $X_n^+ = D(U_n^+)$ and $X_n^- = D(U_n^-)$, then μ_X and σ_X^2 will denote their common mean and common variance, in notational agreement with (2.2).

Using simple properties of modulo-1 addition, we can equivalently represent U_n in (3.1) as

$$U_n^+ \;=\; \langle U_0 \;+\; S_n^V \rangle, \;\; n \geq 0 \tag{3.3}$$

where

$$S_n^V \;=\; \sum_{j=1}^{n} V_j, \;\; n \geq 0 \tag{3.4}$$

is the sum of n i.i.d. variates ($S_0^V = 0$). From (3.2), the analogous representation of U_n^- is

$$U_n^- \;=\; \begin{cases} \langle U_0 \;+\; S_n^V \rangle, & n \textit{ even} \\ \\ \langle 1 \;-\; U_0 \;-\; S_n^V \rangle, & n \textit{ odd} \end{cases} \tag{3.5}$$

Letting f_V denote the common density function of the innovations V_n, it follows that the density function of S_n^V, is the n-fold convolution $f_V^{(n)}$ of f_V. We denote by $f^{(0)}$ the Dirac function $\delta(x)$ which constitutes the identity under convolution, so $\tilde{f}_V^{(n)}(s) = \tilde{f}_V^n(s)$. The sample path relationship of Eq. (3.2) relating $\{U_n^+\}$ to $\{U_n^-\}$ will reduce the amount of work needed to analyze the latter, by deducing many of its properties from the former. For example, Eq. (3.2) implies the sample path relation $\{U_n^-\}_{n=0}^{\infty} = \{U_0, 1 - U_1^+, U_2^+, 1 - U_3^+, \ldots\}$ leading immediately to the identities

$$\psi_U^-(\tau) \;=\; (-1)^\tau \psi_U^+(\tau), \;\; \rho_U^-(\tau) \;=\; (-1)^\tau \rho_U^+(\tau), \;\; \tau = 0, 1, \ldots \tag{3.6}$$

As mentioned before, although Lemma 1 allows the innovations V_n to have any distribution, it is clear that computational and analytical considerations will steer us to simple innovations. Two types of innovations will now be reviewed. Melamed (1991) treats the classes TES$^+(L, R)$ and TES$^-(L, R)$ of simple TES methods with elementary 2-parameter uniform innovations

$$V_n = L + (R - L)Z_n \tag{3.7}$$

where $-\frac{1}{2} \leq L < R < \frac{1}{2}$ and $\{Z_n\}$ is a sequence of i.i.d. Uniform(0,1) variates. The (L, R) parameterization is equivalent to the (α, ϕ) parameterization given by

$$\alpha \;=\; R - L, \;\; \phi \;=\; \begin{cases} \frac{R + L}{\alpha}, & \alpha > 0 \\ \\ 0, & \alpha = 0 \end{cases} \tag{3.8}$$

and the corresponding TES classes are denoted TES$^+(\alpha, \phi)$ and TES$^-(\alpha, \phi)$, respectively. With this notation, the density function of V_n in (3.7) is just

$$f_V(x) \;=\; \frac{1}{\alpha} \, 1_{[L,R)}(x), \tag{3.9}$$

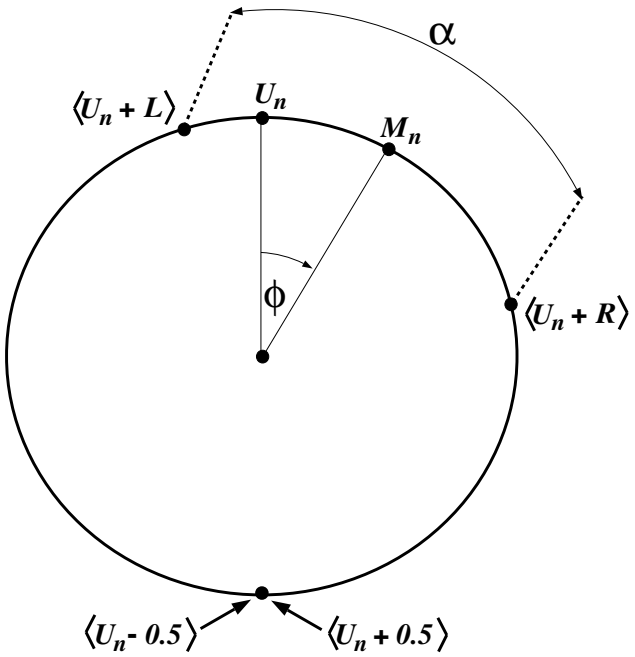

Figure 2: Geometric Interpretation of Elementary TES processes

and its Laplace Transform is

$$\tilde{f}_V(s) \;=\; \frac{e^{-sL} - e^{-sR}}{s\alpha} \;=\; \frac{e^{s\alpha(1-\phi)/2} - e^{-s\alpha(1+\phi)/2}}{s\alpha} \tag{3.10}$$

Figure 2 provides a simple geometric interpretation of TES(α, ϕ) methods as random walks on the unit circle (unit circumference) with random step size V_n. Here, $\langle U_n + L\rangle$ and $\langle U_n + R\rangle$ are translations of the current TES iterate U_n by offsets L and R on the unit circle, α is the length of the circular interval $C(U_n, L, R)$ defined in (2.1) and consisting of all points on the unit circle from $\langle U_n + L\rangle$ and clockwise to $\langle U_n + R\rangle$ (the arc marked as having length α), and ϕ is an indication of the rotation of that interval away from symmetric straddle of U_n (i.e., of the angle between U_n and the interval midpoint M_n). For TES$^+(\alpha, \phi)$, the recursive construction of U^+_{n+1} is achieved by first constructing the circular interval $C(U_n, L, R)$ (the *Expand* stage of TES), and then sampling uniformly in $C(U_n, L, R)$ to obtain the next iterate U_{n+1} (the *Sample* stage of TES). The *Transform* stage in TES$^+$ is null (the identity). With this interpretation, the one-step transition density of $\{U^+_n\}$ on its $[0, 1)$ support is

$$\frac{\partial}{\partial y} P\{U^+_{n+1} \le y \,|U^+_n = x\} \;=\; \frac{1}{\alpha}\, 1[y \;\in\; C(x, L, R)]. \tag{3.11}$$

For TES$^-(\alpha, \phi)$, the recursive construction of U^-_{n+1} is a bit more involved (see Melamed (1991)). The *Transform* stage maps U_n to $1 - U_n$. The *Expand* stage is step dependent: for odd n we take $C(1-U_n, L, R)$, but for even n we take $C(1-U_n, R, L)$ with the offsets reversed. The *Sample* stage to obtain U^-_{n+1} is the same as in TES$^+(\alpha, \phi)$. The one-step transition density of $\{U^-_n\}$ is similar to (3.11) with x replaced by $1-x$, and L and R alternate in the obvious step-dependent manner.

The behavior of TES processes as a function of α and ϕ can be readily grasped with the aid of Figure 2. As $\alpha \downarrow 0$, the limiting circular intervals shrink to a singleton set by (2.1); Consequently, $\{U^+_n\}$ approaches $\{U_0, U_0, \ldots\}$ with $\rho^+_U(\tau) \equiv 1$, while $\{U^-_n\}$ approaches $\{U_0, 1\text{–}U_0, U_0, 1\text{–}U_0, \ldots\}$, with $\rho^-_U(\tau) \equiv -1$. As $\alpha \uparrow 1$, the limiting circular intervals expand to $[0, 1)$ by (2.1), and both $\{U^+_n\}$ and $\{U^-_n\}$ approach i.i.d. Uniform(0,1) variates, so $\rho^+_U(\tau) \equiv \rho^-_U(\tau) \equiv 0$. In between these extremal cases lie intermediate values $\alpha \in (0, 1)$ which give rise to c.i.d. TES processes, such that $\rho^+_U(1)$ covers $(0, 1)$ and $\rho^-_U(1)$ covers $(-1, 0)$ (see Melamed(1991)). Further, $\phi = 0$ means that the next TES iterate U_{n+1} is equally likely to fall to the left of or to the right of the current iterate U_n or the transformed iterate $1-U_n$. In this case, $E[V_n] = 0$, and the random walk has zero drift around the circle. If $\phi > 0$, we have $E[V_n] > 0$, and the drift is clockwise, resulting in cyclical sample paths. The case $\phi < 0$ is analogous but with counter-clockwise drift.

Figure 3 depicts a simple but considerable generalization of the elementary innovations (3.7). The generalization specifies V_n as a mixture of K uniform variates on $[L_k, R_k)$ such that $\frac{1}{2} \leq L_k < R_k < \frac{1}{2}$, with mixing probability P_k, namely,

$$V_n \;=\; \sum_{k=1}^{K} 1_{[C_{k-1}, C_k)}(Z_n)\,[L_k + (R_k - L_k)\,Z_n] \tag{3.12}$$

where $\{C_k\}_{k=0}^K$ is the cumulative distribution function of $\{P_k\}_{k=1}^K$, that is, $C_k = \sum_{j=1}^k P_j$ ($C_0 = 0$ and $C_K = 1$). The attendant density is the step function

$$f_V(x) \;=\; \sum_{k=1}^{K} \frac{P_k}{R_k - L_k}\, 1_{[L_k R_k)}(x) \tag{3.13}$$

and the attendant Laplace Transform is the corresponding mixture with indexed components of the form (3.10). The 3-dimensional representation in Figure 3 depicts a step function density (3.13) whose steps are drawn on the surface of an

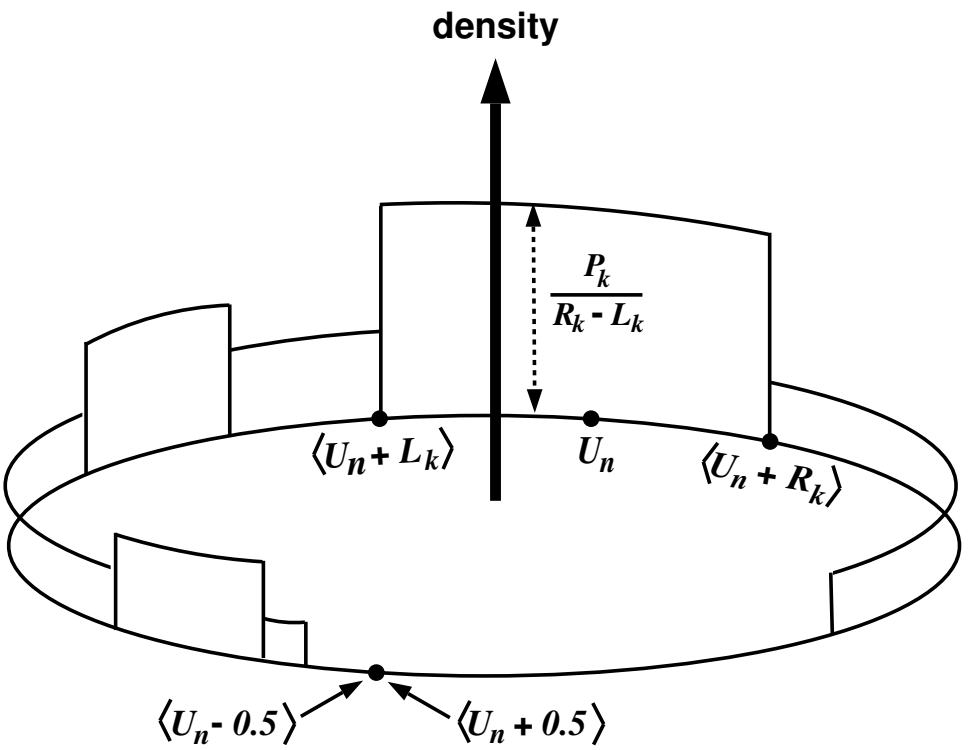

Figure 3: Three-Dimensional Representation of a TES Step Function Density

imaginary cylinder; the cylinder's base rests on the unit circle. The merit of this generalization is due to the fact that mixtures of uniform densities can be used to approximate general densities arbitrarily closely.

It is important to realize that Lemma 1 ensures that the sequences defined by (3.1) and (3.2) possess marginal uniformity for *any* innovation density f_V. However, different choices of f_V will yield a variety of sample path realizations and autocorrelation functions. The simulation study of TES$^+$ (α, ϕ) and TES$^-$ (α, ϕ) methods in Melamed (1991) found that sample paths of TES processes exhibit visual "discontinuities" arising from "crossing" the point 0 on the circle in either direction. Furthermore, for $\phi \neq 0$, the sample paths have a cyclical sawtooth pattern. It was further shown there that sample paths of TES processes with Uniform$(0, 1)$ marginals can be transformed to yield "continuous-looking" realizations, and that cyclical sample paths can be skewed as desired. To this end, one employs a family of simple piecewise linear transformations S_ξ, dubbed *stitching transformations*,

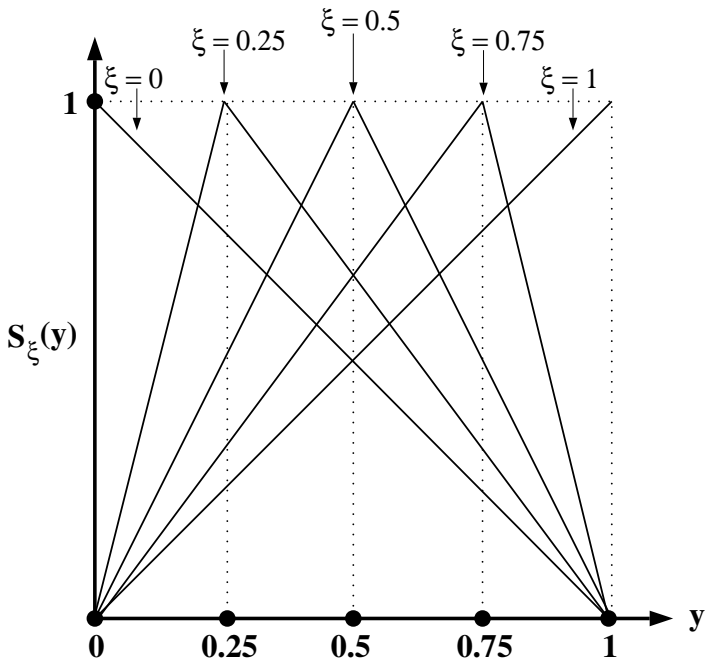

Figure 4: Examples of Stitching Transformations

where

$$S_\xi(y) \;=\; \begin{cases} \dfrac{y}{\xi}, & 0 \le y < \xi \\ \dfrac{1-y}{1-\xi}, & \xi \le y < 1 \end{cases} \tag{3.14}$$

and the S_ξ are parameterized by the *stitching parameter* $\xi \in [0,1]$. It is easily shown (ibid.) that each S_ξ preserves uniformity, and so stitched TES processes of the form $\{S_\xi(U_n)\}$ are also marginally Uniform$(0,1)$ and can be further distorted to have arbitrary marginals. More precisely, for any distortion D and $\xi \in [0,1)$, we define *stitched distortions* D_ξ of the form

$$D_\xi(x) \;=\; D(S_\xi(x)). \tag{3.15}$$

Figure 4 displays a number of stitching transformations S_ξ, for a variety of stitching parameters ξ. Note that for $0 < \xi < 1$, each S_ξ is continuous on the circle in the sense that $S_\xi(0) = S_\xi(1-)$. This fact renders $\{S_\xi(U_n)\}$, and consequently $\{D_\xi(U_n)\}$, more "continuous-looking" than their unstitched counterparts. Figure 4 suggests how the choice of ξ determines a skewing value for cyclical TES processes;

in particular, $\xi = 0.5$ gives rise to symmetric cycles, whereas $0 < \xi < 0.5$ give rise to cycles skewed to the left, and $0.5 < \xi < 1$ give rise to cycles skewed to the right. Note that the case $\xi = 1$ corresponds to no stitching (S_1 is the identity), while $\xi = 0$ corresponds to the antithetic transformation $S_0(y) = 1 - y$, both of which are not continuous on the unit circle. The case $\xi = 0.5$ is special. It is easily verified from (3.14) that $S_{0.5}(y) = S_{0.5}(1-y)$ for all $y \in [0, 1)$. Consequently, Eqs. (3.1)-(3.2) readily imply that the sample paths of $\{S_{0.5}(U_n^+)\}$ and $\{S_{0.5}(U_n^-)\}$ are identical, whence the processes are stochastically equivalent. Empirical evidence shows that as ξ moves away from 0.5 in either direction, the discrepancy between the sample paths of $\{S_\xi(U_n^+)\}$ and $\{S_\xi(U_n^-)\}$ gradually increases, and the absolute value of the attendant autocorrelation functions decreases concomitantly.

4 TRANSITION STRUCTURE OF TES SEQUENCES

Let $g_\tau^+(y|x)$ denote the conditional density of U_τ^+ given $U_0 = x$, and let $g_\tau^-(y|x)$ denote the conditional density of U_τ^- given $U_0 = x$. In the next two theorems we derive these conditional densities; note that we make no assumptions on the innovation density f_V, other than the mild conditions which enable (2.3).

Theorem 1 *$\{U_n^+\}$ is a stationary Markov process with transition density $g_\tau^+(y|x)$, where for $\tau = 0, 1, \ldots$*

$$g_\tau^+(y|x) = \begin{cases} \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, e^{i2\pi\nu(y-x)}, & 0 \le y, x < 1 \\ 0, & \textit{otherwise} \end{cases} \tag{4.1}$$

or in real form,

$$g_\tau^+(y|x) = \begin{cases} 1 + 2\sum_{\nu=1}^{\infty} Re[\tilde{f}_V^\tau(i2\pi\nu)\, e^{i2\pi\nu(y-x)}], & 0 \le y, x < 1, \\ 0, & \textit{otherwise} \end{cases} \tag{4.2}$$

Proof $\{U_n^+\}$ is stationary Markovian by construction; see Eq. (3.1). For $\tau = 0$, both Eqs. (4.1) and (4.2) reduce to the Dirac function $\delta(y-x)$. For $\tau > 0$ we can

write by virtue of (3.3),

$$\tilde{g}_\tau^+(s|x) \;=\; E\Big[e^{-sU_\tau^+}|U_0=x\Big] \;=\; E\Big[e^{-s\langle x+\,S_\tau^V\rangle}\Big]$$

$$=\; \int_{-\infty}^{\infty} e^{-s\langle x+\sigma\rangle} f_V^{(\tau)}(\sigma)\,d\sigma \;=\; \sum_{\nu=-\infty}^{\infty}\int_{\nu}^{\nu+1} e^{-s\langle x+\sigma\rangle} f_V^{(\tau)}(\sigma)\,d\sigma.$$

Since $e^{-s\langle x+\sigma\rangle}$ is periodic in σ (period 1), we can rewrite the above as

$$\tilde{g}_\tau^+(s|x) \;=\; \sum_{\nu=-\infty}^{\infty}\int_0^1 e^{-s\langle x+\sigma\rangle} f_V^{(\tau)}(\sigma+\nu)\,d\sigma$$

$$=\int_0^1 e^{-s\langle x+\sigma\rangle}\Big[\sum_{\nu=-\infty}^{\infty} f_V^{(\tau)}(\sigma+\nu)\Big]d\sigma \;=\; \sum_{\nu=-\infty}^{\infty}\tilde{f}_V^\tau(i2\pi\nu)\int_0^1 e^{-s\langle x+\sigma\rangle}+i2\pi\nu\sigma d\sigma$$

where the interchange of summation and integration in the second equality is justified by uniform convergence, and in the third equality because a Fourier Series can be integrated termwise (Bary (1964), Vol. 1, Chapter 1); note also that we used the Poisson summation formula (2.3) in the last equality. Since $0 \le x < 1$, we can decompose the last integral as follows

$$\tilde{g}_\tau^+(s|x) \;=\; \sum_{\nu=-\infty}^{\infty}\tilde{f}_V^\tau(i2\pi\nu)\Big[\int_0^{1-x} + \int_{1-x}^1\Big].$$

Now,

$$\int_0^{1-x} \;=\; \int_0^{1-x} e^{-s(x+\sigma)+i2\pi\nu\sigma}\,d\sigma \;=\; \frac{e^{-s-i2\pi\nu x}-e^{-sx}}{-s+i2\pi\nu}$$

$$\int_{1-x}^1 \;=\; \int_{1-x}^1 e^{-s(x+\sigma-1)+i2\pi\nu\sigma}\,d\sigma \;=\; \frac{e^{-sx}-e^{-i2\pi\nu x}}{-s+i2\pi\nu}$$

whence, after adding the integrals,

$$\tilde{g}_\tau^+(s|x) \;=\; \sum_{\nu=-\infty}^{\infty} e^{-i2\pi\nu x}\,\frac{\tilde{f}_V^\tau(i2\pi\nu)}{s-i2\pi\nu} - e^{-s}\sum_{\nu=-\infty}^{\infty} e^{-i2\pi\nu x}\,\frac{\tilde{f}_V^\tau(i2\pi\nu)}{s-i2\pi\nu}$$

$$=\; \tilde{c}_\tau(s|x)-\tilde{d}_\tau(s|x).$$

we next observe that $\tilde{e}(s) = \dfrac{1}{s-i2\pi\nu}$ is the Laplace Transform of the function $e(y) = e^{i2\pi\nu y}\,1_{[0,\infty)}(y)$, so inverting $\tilde{c}_\tau(s|x)$ yields

$$c_\tau(y|x) \;=\; \sum_{\nu=-\infty}^{\infty} e^{i2\pi\nu(y-x)}\,\tilde{f}_V^\tau(i2\pi\nu)\,1_{[0,\infty)}(y).$$

Now, the effect of e^{-s} in $\tilde{d}_\tau(s|x)$ is translation by 1, so inverting $\tilde{d}_\tau(s|x)$ yields

$$\begin{aligned} d_\tau(y|x) &= \sum_{\nu=-\infty}^{\infty} e^{i2\pi\nu(y-1-x)}\, \tilde{f}_V^\tau(i2\pi\nu)\, 1_{[1,\infty)}(y) \\ &= \sum_{\nu=-\infty}^{\infty} e^{i2\pi\nu(y-x)}\, \tilde{f}_V^\tau(i2\pi\nu)\, 1_{[1,\infty)}(y), \end{aligned}$$

by the periodicity of the complex exponential. Eq. (4.1) now follows by calculating $g_\tau^+(y|x) = c_\tau(y|x) - d_\tau(y|x)$ from the expressions above. Finally, Eq. (4.2) follows from Eq. (4.1) by noting that the term for $\nu = 0$ evaluates to 1; furthermore, terms indexed by $-\nu$ and ν are complex conjugates by Lemma 2, so summing them yields twice their real part. □

Theorem 2 *$\{U_n^-\}$ is a stationary Markov process with transition density $g_\tau^-(y|x)$, where for $\tau = 0, 1, \ldots$*

$$g_\tau^-(y|x) = \begin{cases} \displaystyle\sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, e^{i2\pi\nu(y-x)}, & 0 \le y, x < 1, \quad \tau \textit{ even} \\ \displaystyle\sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, e^{-i2\pi\nu(y+x)}, & 0 \le y, x < 1, \quad \tau \textit{ odd} \\ 0, \quad \textit{otherwise} \end{cases} \tag{4.3}$$

or in real form,

$$g_\tau^-(y|x) = \begin{cases} 1 + 2\displaystyle\sum_{\nu=1}^{\infty} Re[\tilde{f}_V^\tau(i2\pi\nu)\, e^{i2\pi\nu(y-x)}], & 0 \le y, x < 1, \quad \tau \textit{ even} \\ 1 + 2\displaystyle\sum_{\nu=1}^{\infty} Re[\tilde{f}_V^\tau(i2\pi\nu)\, e^{-i2\pi\nu(y+x)}], & 0 \le y, x < 1, \quad \tau \textit{ odd} \\ 0, \quad \textit{otherwise} \end{cases} \tag{4.4}$$

Proof $\{U_n^-\}$ is stationary Markovian by construction; see Eq. (3.2). Directly from Eq. (3.2) we have

$$g_\tau^-(y|x) = \begin{cases} g_\tau^+(y|x), & \tau \textit{ even} \\ g_\tau^+(1-y|x), & \tau \textit{ odd} \end{cases}$$

Since the case of even τ was already computed in Theorem 1, it remains to prove (4.3) for the case of odd τ. Substituting $1-y$ for y in (4.1) yields

$$g_\tau^-(y\,|x) \;=\; \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, e^{i2\pi\nu(1-y-x)}$$

which is equivalent to Eq. (4.3) by the periodicity of the complex exponential. Finally, Eq. (4.4) follows from Eq. (4.3) by an argument analoguous to that of Theorem 1. □

Theorems 1 and 2 readily specialize to the classes of TES$^+(\alpha,\phi)$ and TES$^-(\alpha,\phi)$ with the elementary innovations (3.7).

Corollary 1 *For TES$^+(\alpha,\phi)$ and TES$^-(\alpha,\phi)$, the corresponding transition densities are for $\tau = 0, 1, \ldots$*

$$g_\tau^+(y\,|x) = \begin{cases} 1 + 2\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu\alpha)}{\pi\nu\alpha}\right)^\tau \cos(\pi\nu\alpha\phi\tau - 2\pi\nu(y-x)), & 0 \le y, x < 1 \\ 0, \quad \textit{otherwise} \end{cases} \tag{4.5}$$

$$g_\tau^-(y\,|x) \;=\; \begin{cases} 1 + 2\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu\alpha)}{\pi\nu\alpha}\right)^\tau \cos(\pi\nu\alpha\phi\tau - 2\pi\nu(y-x)), & 0 \le y, x < 1, \\ & \tau \textit{ even} \\ 1 + 2\sum_{\nu=1}^{\infty} \left(\frac{\sin(\pi\nu\alpha)}{\pi\nu\alpha}\right)^\tau \cos(\pi\nu\alpha\phi\tau + 2\pi\nu(y+x)), & 0 \le y, x < 1, \\ & \tau \textit{ odd} \\ 0, \quad \textit{otherwise} \end{cases} \tag{4.6}$$

Proof From Eq. (3.10), we have

$$\begin{aligned} \tilde{f}_V^\tau(i2\pi\nu) \;&=\; \left(\frac{e^{i\pi\nu\alpha(1-\phi)} - e^{-i\pi\nu\alpha(1+\phi)}}{i2\pi\nu\alpha}\right)^\tau \\ &=\; \left(\frac{e^{-i\pi\nu\alpha\phi}}{\pi\nu\alpha}\right)^\tau \left(\frac{e^{i\pi\nu\alpha} - e^{-i\pi\nu\alpha}}{2i}\right)^\tau = \left(\frac{\sin(\pi\nu\alpha)}{\pi\nu\alpha}\right)^\tau e^{-i\pi\nu\alpha\phi\tau} \end{aligned} \tag{4.7}$$

where the last equality follows from Euler's Formula for the sine function. The rest follows by substituting (4.7) into (4.2) and (4.4), respectively. It can be checked via

moderately tedious algebra that the represntations (4.5) and (3.11) of $g_1^+(y|x)$ are equivalent, and similarly for $g_1^-(y|x)$. □

It is of interest to compare TES processes to their time-reversed counterparts which are also stationary Markovian (see Kelly (1979), Lawrance (1991)). Let $\{U_n^+\}_{n=0}^{\tau}$ be a TES$^+$ subsequence and define its time-reversed version $\{\hat{U}_n^+\}_{n=0}^{\tau}$ by $\hat{U}_n^+ = U_{\tau-n}^+$. Now, from Eq. (3.4)

$$U_\tau^+ = \langle U_0 + S_\tau^V \rangle. \tag{4.8}$$

Using modulo-1 arithmetic we can isolate U_0 as $U_0 = \langle U_\tau^+ - S_\tau^V \rangle$, or equivalently in terms of the time-reversed subsequence

$$\hat{U}_\tau^+ = \langle \hat{U}_0^+ - S_\tau^V \rangle. \tag{4.9}$$

A comparison of (4.8) and (4.9) reveals that a time-reversed TES$^+$ sequence is also a TES$^+$ sequence with an innovation sequence $\{\hat{V}_n\}$, where $\hat{V}_n = -V_{\tau-n}$. We refer to the $\hat{V}_n$ as *reverse innovations*. Indeed, the transition density $\hat{g}_\tau^+$ of $\{\hat{U}_n^+\}$ is related to $g_\tau^+(y|x)$ by $\hat{g}_\tau^+(y|x) = g_\tau^+(x|y)$ on their common support, because the marginals of $\{U_n^+\}$ and $\{\hat{U}_n^+\}$ are each uniform on $[0,1)$.

Next, consider the time-reversed subsequence $\{\hat{U}_n^-\}_{n=0}^{\tau}$ defined by $\hat{U}_n^- = U_{\tau-n}^-$. For even τ, the situation is analoguous to TES$^+$ since the even-index TES$^-$ subsequence is an even-index TES$^+$ subsequence. For odd τ, the situation is quite different. In fact, from Eq. (3.5)

$$U_\tau^- = \langle -U_0 - S_\tau^V \rangle, \quad \tau \ odd, \tag{4.10}$$

so that $U_0 = \langle -U_\tau^- - S_\tau^V \rangle$, or equivalently in terms of the time-reversed subsequence,

$$\hat{U}_\tau^- = \langle -\hat{U}_0 - S_\tau^V \rangle, \quad \tau \ odd. \tag{4.11}$$

Clearly, the representations (4.10) and (4.11) are identical. Therefore, we reach the somewhat surprising conclusion that the odd-index subsequence of a TES$^-$ process is time-reversible in the sense that the transition density from U_0 to U_τ^- is identical to the transition density from U_τ^- to U_0. Indeed, from Eq. (4.3), $g_\tau^-(y|x) = g_\tau^-(x|y)$ for odd τ, which confirms the reversibility of any odd-index TES$^-$ subsequence. Notice that for TES$^-$, $\hat{V}_n = V_{\tau-n}$, so $\{\hat{V}_n\}$ has the same probability law as $\{V_n\}$ (permuting the indices of an i.i.d. sequence has no effect on its joint distribution).

Consider now the special case of the uniform innovations (3.7), where the reverse innovations become

$$\hat{V}_n \;=\; -L - \alpha Z_{\tau-n} \;=\; \alpha - R - \alpha Z_{\tau-n} \;=\; -R + \alpha(1 - Z_{\tau-n}). \qquad (4.12)$$

Since $\{1-Z_n\}$ is an i.i.d. Uniform(0,1) sequence, we see from (4.12) that the effect of reverse innovations is to change TES$^+(L, R)$ to TES$^+(-R, -L)$; equivalently, TES$^+(\alpha, \phi)$ under time reversal becomes TES$^+(\alpha, -\phi)$ by virtue of (3.8). The odd-index TES$^-(\alpha, \phi)$ subsequence is of course reversible. An examination of Eqs. (4.5) and (4.6) reveals that $\{U_n^+\}$ and $\{U_n^-\}$ have transition densities which are symmetric in their arguments x and y if and only if $\phi\alpha$ is an integer. If we exclude the trivial cases $\alpha = 0$ and $\alpha = 1$, we are led to conclude that $\phi = 0$ is necessary and sufficient for reversibility of TES$^+(\alpha, \phi)$ and TES$^-(\alpha, \phi)$. This accords with the geometric interpretation of TES as a random walk on a circle. For $\phi = 0$, TES(α, ϕ) becomes a driftless reversible process, while for $\phi \neq 0$ the attendant drift renders TES(α, ϕ) a directional process.

5 AUTOCOVARIANCE OF DISTORTED TES SEQUENCES

Consider the distorted TES sequences $\{X_n^+\}_{n=0}^\infty$ and $\{X_n^-\}_{n=0}^\infty$ where $X_n^+ = D(U_n^+)$ and $X_n^- = D(U_n^-)$ are assumed to have a finite second moment. In this section we derive the autocovariance function $\psi_X^+(\tau)$ of $\{X_n^+\}$ and $\psi_X^-(\tau)$ of $\{X_n^-\}$ with the aid of Theorems 1 and 2.

Theorem 3 *For* $\tau \;=\; 0, 1, \ldots$

$$\psi_X^+(\tau) \;=\; \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\,|\tilde{D}(i2\pi\nu)|^2 \qquad (5.1)$$

or in real form,

$$\psi_X^+(\tau) \;=\; 2\sum_{\nu=1}^{\infty} Re[\tilde{f}_V^\tau(i2\pi\nu)]\,|\tilde{D}(i2\pi\nu)|^2 \qquad (5.2)$$

Proof By definition,

$$\psi_X^+(\tau) \;=\; E[X_0^+ X_\tau^+] - \mu_X^2 \;= \int_0^1 \int_0^1 D(x)\, D(y)\, g_\tau^+(y|x)\, dx\, dy - \left[\int_0^1 D(x)\, dx\right]^2$$

where in the first integral we used the fact that the conditional density $g_\tau^+(y|x)$ is also the joint density of U_0 and U_τ. Substituting the representation of $g_\tau^+(y\,|x)$ from (4.1) gives

$$\begin{aligned} E[X_0^+ X_\tau^+] \;&=\; \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 \int_0^1 e^{i2\pi\nu(y-x)}\, D(x)\, D(y)\, dx\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 e^{-i2\pi\nu x}\, D(x)\, dx \int_0^1 e^{i2\pi\nu y}\, D(y)\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, \tilde{D}(i2\pi\nu)\, \tilde{D}(-i2\pi\nu). \end{aligned}$$

From Lemma 2, $\tilde{D}(i2\pi\nu)$ and $\tilde{D}(-i2\pi\nu)$ are complex conjugates, so their product is $|\tilde{D}(i2\pi\nu)|^2$. We further note that the term indexed by $\nu = 0$ in the infinite sum above is

$$\tilde{f}_V^\tau(0) \int_0^1 D(x)\, dx \int_0^1 D(y)\, dy \;=\; \mu_X^2$$

which establishes Eq. (5.1). Finally, Eq. (5.2) follows from Eq. (5.1) and Lemma 2 by noting that terms indexed by $-\nu$ and ν are complex conjugates. □

Theorem 4 *For* $\tau = 0, 1, \ldots$

$$\psi_X^-(\tau) \;=\; \begin{cases} \displaystyle\sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, |\tilde{D}(i2\pi\nu)|^2, & \tau \textit{ even} \\[2ex] \displaystyle\sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, Re[\tilde{D}^2(i2\pi\nu)], & \tau \textit{ odd} \end{cases} \tag{5.3}$$

or in real form,

$$\psi_X^-(\tau) \;=\; \begin{cases} 2\displaystyle\sum_{\nu=1}^{\infty} Re[\tilde{f}_V^\tau(i2\pi\nu)]\, |\tilde{D}(i2\pi\nu)|^2, & \tau \textit{ even} \\[2ex] 2\displaystyle\sum_{\nu=1}^{\infty} Re[\tilde{f}_V^\tau(i2\pi\nu)]\, Re[\tilde{D}^2(i2\pi\nu)], & \tau \textit{ odd} \end{cases} \tag{5.4}$$

Proof The case of even τ was already calculated in Theorem 3 by virtue of Eq. (3.2). For the case of odd τ, we first calculate $E[X_0^- X_\tau^-]$. To this end, it suffices to substitute the odd τ case of $g_\tau^-(y|x)$ from Eq. (4.3) for $g_\tau^+(y|x)$ everywhere in the proof of Theorem 3. We get

$$\begin{aligned} E[X_0^- X_\tau^-] &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 \int_0^1 e^{-i2\pi\nu(y+x)}\, D(x)\, D(y)\, dx\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 e^{-i2\pi\nu x}\, D(x)\, dx \int_0^1 e^{-i2\pi\nu y}\, D(y)\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, \tilde{D}^2(i2\pi\nu). \end{aligned} \tag{5.5}$$

Next, we observe the identities

$$E[X_0^- X_\tau^-] \;=\; E[X_1^- X_{\tau+1}^-] \;=\; E[D(1-U_1^+)\, D(U_{\tau+1}^+)],$$

where the first identity follows from the stationarity of $\{X_n^-\}$, and the second is a consequence of (3.2) and the fact that $\tau + 1$ is even by assumption. The second identity shows that to calculate the right hand side above, it suffices to substitute $D(1-x)$ for $D(x)$ and to use $g_\tau^+(y|x)$ from Eq. (4.1) in a calculation analoguous to (5.5) . We get

$$\begin{aligned} E[X_1^- X_{\tau+1}^-] &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 \int_0^1 e^{i2\pi\nu(y-x)}\, D(1-x)\, D(y)\, dx\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 \int_0^1 e^{i2\pi\nu(y+x)}\, D(x)\, D(y)\, dx\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \int_0^1 e^{i2\pi\nu x}\, D(x)\, dx \int_0^1 e^{i2\pi\nu y}\, D(y)\, dy \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu) \tilde{D}^2(-i2\pi\nu). \end{aligned} \tag{5.6}$$

Combining the representations (5.5) and (5.6) for $E[X_0^- X_\tau^-]$ yields

$$\begin{aligned} E[X_0^- X_\tau^-] &= \frac{1}{2}\left(E[X_0^- X_\tau^-] + E[X_1^- X_{\tau+1}^-]\right) \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, \frac{1}{2}\,[\tilde{D}^2(i2\pi\nu) + \tilde{D}^2(-i2\pi\nu)]. \end{aligned}$$

But Lemma 2 implies that $\tilde{D}^2(i2\pi\nu)$ and $\tilde{D}^2(-i2\pi\nu)$ are complex conjugates, so their sum is twice their real part. Hence,

$$E[X_0^- X_\tau^-] \;= \sum_{\nu=-\infty}^{\infty} \tilde{f}_V^\tau(i2\pi\nu)\, Re[\tilde{D}^2(i2\pi\nu)],$$

which establishes (5.3). Eq. (5.4) now follows similarly to the proof of Theorem 3, since the term indexed by $\nu = 0$ equals μ_X^2, and terms indexed by $-\nu$ and ν are complex conjugates by Lemma 2. □

Remark 1 As a check, set $\tau = 0$ in Eqs. (5.1) and (5.3), to obtain

$$\psi_X^+(0) \;=\; \psi_X^-(0) \;=\; \sum_{\nu=-\infty}^{\infty} |\tilde{D}(i2\pi\nu)|^2 - \mu_X^2 \;=\; \int_0^1 D^2(x)\,dx - \mu_X^2 \;=\; \sigma_X^2$$

where we used Parseval's Equality (Bary (1964)) in the second equality. This yields the variance as required. □

Remark 2 An interesting consequence of Theorems 3 and 4 is that replacing the innovations V_n by their reverse counterparts $\hat{V}_n = -V_{\tau-n}$ leaves the autocorrelation structure unchanged. To see that, note that the effect of reverse innovations is to replace $\tilde{f}_V^\tau(i2\pi\nu)$ by $\tilde{f}_V^\tau(-i2\pi\nu)$ everywhere in Theorems 3 and 4. But this substitution leaves Eqs. (5.2) and (5.4) unchanged, because $Re[\tilde{f}_V^\tau(-i2\pi\nu] = Re[\tilde{f}_V^\tau(i2\pi\nu)]$ (from Lemma 2 we take the real part of complex conjugates). In particular, recall that for the uniform innovations (3.7), reverse innovations change TES$^+(L, R)$ to TES$^+(R, L)$. But Eq. (3.8) implies that this is equivalent to replacing ϕ by $-\phi$. We conclude that TES$^+(\alpha, \phi)$ and TES$^+(\alpha, -\phi)$ have the same autocovariance function $\psi_X^+(\tau)$; similarly, TES$^-(\alpha, \phi)$ and TES$^-(\alpha, -\phi)$ have the same autocovariance function $\psi_X^-(\tau)$. The transition structure (and the sample paths) are, of course, quite different, except for the (reversible) odd-index TES$^-(\alpha, \phi)$ subsequences. □

Remark 3 Theorems 1-4 can be modified to accommodate non-stationary TES sequences. If the innovations $\{V_n\}$ are non-stationary, then Eqs. (5.1)-(5.4) will still hold provided $\tilde{f}_V^\tau$ is replaced by the τ-fold convolution $\tilde{f}_V^{(\tau)}$. If, however, we replace D by a distortion sequence $\{D^{(n)}\}$, then Eqs. (5.1) and (5.3) must be modified. Specifically, one must replace $|\tilde{D}(i2\pi\nu)|^2$ by $\tilde{D}^{(0)}(i2\pi\nu)\tilde{D}^{(\tau)}(-i2\pi\nu)$ and $\tilde{D}^2(i2\pi\nu)$ by $\tilde{D}^{(0)}(i2\pi\nu)\tilde{D}^{(\tau)}(i2\pi\nu)$. However, the real form of Eqs. (5.2) and (5.4) can no longer be deduced, as their proof depends heavily on stationarity. □

We conclude this section by obtaining a result which would enable us to relate distorted stitched TES processes to their unstitched TES counterparts, for any distortion D.

Theorem 5 *For any* $0 \leq \xi \leq 1$,

$$\tilde{D}_\xi(s) \;=\; \xi\,\tilde{D}(\xi s) + (1-\xi)\,e^{-s}\,\tilde{D}(-(1-\xi)s). \tag{5.7}$$

Proof By definition,

$$D_\xi(s) \;= \int_0^1 e^{-sx}\,D(S_\xi(x))\,dx \;= \int_0^\xi + \int_\xi^1$$

where by Eq. (3.14)

$$\int_0^\xi \;=\; \int_0^\xi e^{-sx}\,D\!\left(\frac{x}{\xi}\right)dx \;= \int_0^1 e^{-s\xi y}\,D(y)\,\xi\,dy \;=\; \xi\,\tilde{D}(s\xi)$$

$$\int_\xi^1 \;=\; \int_\xi^1 e^{-sx}\,D\!\left(\frac{1-x}{1-\xi}\right)dx \;= \int_0^1 e^{-s}\,e^{-s(-(1-\xi))y}\,D(y)\,(1-\xi)\,dy$$

$$=\; (1-\xi)\,e^{-s}\,\tilde{D}(-s(1-\xi)).$$

The result follows by adding the integrals above. □

Remark 4 Theorems 3 and 4 remain valid when we replace $\tilde{D}(i2\pi\nu)$ by $\tilde{D}_\xi(i2\pi\nu)$ from Eq. (5.7). Thus, we can obtain the autocorrelations of stitched and distorted TES processes by reference to their unstitched counterparts. □

Corollary 2 *For any integer* ν,

$$\tilde{D}_\xi(i2\pi\nu) \;=\; \xi\,\tilde{D}(i2\pi\nu\xi) + (1-\xi)\,\tilde{D}(-i2\pi\nu(1-\xi)) \tag{5.8}$$

Proof Just substitute $s = i2\pi\nu$ into (5.7), and note that $e^{-i2\pi\nu} = 1$. □

Corollary 3 *For any integer* ν, $\tilde{D}_\xi(i2\pi\nu)$ *and* $\tilde{D}_{1-\xi}(i2\pi\nu)$ *are complex conjugates.*

Proof Substituting $1-\xi$ for ξ in (5.8) yields

$$\tilde{D}_{1-\xi}(i2\pi\nu) \;=\; \xi\,\tilde{D}(-i2\pi\nu\xi) + (1-\xi)\,\tilde{D}(i2\pi\nu(1-\xi)). \tag{5.9}$$

From Lemma 2, $\tilde{D}(i2\pi\nu\xi)$ and $\tilde{D}(-i2\pi\nu\xi)$ are complex conjugates, and so are $\tilde{D}(-i2\pi\nu(1-\xi))$ and $\tilde{D}(i2\pi\nu(1-\xi))$. The rest follows from (5.8) and (5.9) by observing that the complex conjugate property is preserved under addition and under scaling by a real number. □

Let $\{U_n\}$ be any TES process (TES$^+$ or TES$^-$) as defined in Eqs. (3.1) and (3.2). We can now identify an important symmetry for the autocorrelation function of distorted and stitched TES processes $X_\xi = \{X_{\xi,n}\}_{n=0}^\infty$ of the form $X_{\xi,n} = D_\xi(U_n)$.

Theorem 6 *For each* $0 \leq \xi \leq 1$,

$$\rho_{X_\xi}(\tau) \;=\; \rho_{X_{1-\xi}}(\tau), \quad \tau = 0, 1, \ldots \tag{5.10}$$

Proof FromTheorems 3 and 4, the corresponding autocorrelation functions depend on ξ only via the numbers $|\tilde{D}_\xi(i2\pi\nu)|^2$ and $Re[\tilde{D}_\xi(i2\pi\nu)^2]$. But for any real numbers a and b, the complex conjugate numbers $a + ib$ and $a - ib$ satisfy

$$|a+ib|^2 \;=\; |a-ib|^2 \;=\; a^2+b^2, \quad Re[(a+ib)^2] \;=\; Re[(a-ib)^2] \;=\; a^2-b^2 \tag{5.11}$$

The Theorem now follows from Eq. (5.11) and Corollary 3. □

We conclude by pointing out that while the autocorrelation functions ρ_{X_ξ} and $\rho_{X_{1-\xi}}$ in Eq. (5.10) are identical, the sample paths of $\{X_{\xi,n}\}$ and $\{X_{1-\xi,n}\}$ will generally differ, except for $\xi = 0.5$.

6 CONCLUSION

Theorems 3-5 provide a theoretical basis for a practical TES-based modeling methodology for stochastic sequences, which can capture simultaneously both the marginal distribution and the autocorrelation structure of empirical data. For any empirical sample, the marginal density is modeled as an empirical histogram $\hat{h}$ (step function density) from which the distribution function $\hat{H}$ is constructed as a piecewise linear function; $\hat{H}$ is then easily inverted to provide the requisite distortion $D = \hat{H}^{-1}$. The modeling activity consists of a heuristic search for a suitable innovation density, namely, one that gives rise to a stitched and distorted TES process such that the resulting autocorrelation function approximates its empirical counterpart.

Naturally, we need to structure the search and confine it to a suitable subset of all possible innovation densities – a subset which is easy to specify and yet broad enough to provide modeling flexibility. A natural subset enjoying these properties is the class of all step function densities (3.13) with support on $[-0.5, 0.5)$. Notice that this class is dense in the space of all probability densities with the same support in the appropriate topology, yet they are easily specified as finite sequences of triplets of the form (L_k, R_k, P_k). For each innovation (with step function density), Theorems 3-5 can be used to rapidly and accurately compute the attendant TES autocorrelations.

A visual comparison of such numerically computed autocorrelations with their empirical counterparts then determines the next innovation to try, if any. Extensive experimentation has yielded rules of thumb on how TES autocorrelations behave as a function of the innovation density and stitching parameter, and this knowledge has been used to guide the search process. If enough rules can be extracted, there is even hope of partially automating the heurisitic search by using rule-based machine search to identify promising innovation candidates. The final culling would then be done by an analyst who would select the best TES model or models based on the "resemblance" of the empirical sample to simulated realizations generated by candidate TES models. Such resemblance is difficult to quantify, as it calls for heuristic judgment with a human subjective component involving perception and cognition. Nevertheless, it can be delegated to a human modeler, and together with the heuristic search it forms that portion of modeling which is Art rather than strict Science.

We have, in fact, proceeded to build a visual interactive software environment called TEStool which implements the TES modeling methodology sketched above. A detailed description of TEStool appears in Geist and Melamed (1991); a briefer one may be found in Geist and Melamed (1992). TEStool casts the heuristic search process in the mold of an arcade game. The user uses the mouse to visually create/delete and resize innovation density steps, thereby triggering a computation and redisplay of a TES autocorrelation graph superimposed on its empirical counterpart. Visual feedback guides the search process – the goal being to bring the TES autocorrelation graph as close as possible to the corresponding empirical target. This is usually a pleasant, if not amusing, way to do serious modeling work.

We have used the TES methodology and TEStool as a convenient experimentation environment and also to model real-life data. In particular, we modeled the stochastic bit rate process of compressed VBR (*Variable Bit Rate*) video with encouraging results. Our experience includes modeling of empirical frame-level bit rate sequences with monotone autocorrelation function (Lee et al. (1991)), as well as block-level bit rate sequences with oscillatory autocorrelation structure (Melamed et al. (1992)).

The theoretical and practical efficacy of the numerical formulas in Theorems 3-5

will be exhibited in a companion paper (Jagerman and Melamed (1992)) which compares numerically computed TES autocorrelations to the corresponding simulation-based statistics. The speed and accuracy of the formulas will be demonstrated for a wide range of examples with a variety of distortions and innovation densities.

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