

# MARM Processes

## Part I: General Theory

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### Abstract

This paper introduces a new class of real vector-valued stochastic processes, called *MARM* (*Multivariate Autoregressive Modular*) processes, which generalizes the class of (univariate) *ARM* (*Autoregressive Modular*) processes. Like ARM processes, the key advantage of MARM processes is their ability to fit a strong statistical signature consisting of first-order and second-order statistics. More precisely, MARM processes exactly fit an arbitrary multi-dimensional marginal distribution and approximately fit a set of leading autocorrelations and cross-correlations. This capability appears to render the MARM modeling methodology unique in its ability to fit a multivariate model to such a class of strong statistical signatures. The paper describes the construction of two flavors of MARM processes,  $MARM^+$  and  $MARM^-$ , studies the statistics of MARM processes (transition structure and second order statistics), and devises MARM-based fitting and forecasting algorithms providing point estimators and confidence intervals. The efficacy of the MARM fitting and forecasting methodology will be illustrated on real-life data in a companion paper.

**Keywords and Phrases:** MARM Processes, Autoregressive Modular Processes, MARM Modeling Methodology, MARM Forecasting Methodology.

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# 1. Introduction

Consider the following fitting problem: Given a *statistical signature*<sup>1</sup> comprised of an  $n$ -dimensional marginal distribution and an  $n \times n$  set of correlation functions consisting of  $n$  autocorrelation functions and  $n(n-1)/2$  cross-correlation functions, find a probability law (stochastic process model), which satisfies the following goodness-of-fit criteria:

- (a) The multidimensional distribution of the model matches its counterpart in the statistical signature.
- (b) The leading autocorrelations and cross-correlations of the model approximate their counterparts in the statistical signature.
- (c) If the statistical signature is empirically based, then the sample paths generated by Monte Carlo simulation of the model should visually “resemble” the empirical data.

Requirement (a) and (b) represent quantitative goodness-of-fit criteria for capturing first-order and second-order statistics. Those are well-defined requirements, which allow numerical goodness-of-fit comparison across model candidates. In contrast, requirement (c) is qualitative and cannot be defined with mathematical rigor; rather, it is left to human judgment, which is subjective and cognitive. However, modelers routinely attempt to validate their models by visual inspection that ensures “path similarity”, perceived by multiple observers. Consequently, requirement (c) is included as a heuristic goodness-of-fit criterion. We mention that the goodness-of-fit criteria above generalize those in [Jagerman and Melamed (1992a), Melamed (1993, 1999)]. A similar approach was employed in Cario and Nelson (1996).

Note that the problem formulation in this paper does not constrain the source of the statistical signature. But since in practice, a statistical signature is estimated from some empirical time series, we shall henceforth assume, for the sake of concreteness and without loss of generality, that the underlying statistical signature is empirical, that is, computed from an empirical sample. However, we shall not be concerned here with the statistical aspects of such estimation, but focus on the stochastic-processes aspects of the problem. Furthermore, to allow ready estimation of an empirical statistical signature, we require that the underlying empirical multivariate time series,  $\{\vec{X}_k\}_{k=0}^{\infty}$ , be stationary in the sense that the following conditions hold:

1. The marginal distribution of  $\{\vec{X}_k\}_{k=0}^{\infty}$  is time invariant, that is, independent of  $k$ .
2. The covariance matrix of  $\{\vec{X}_k\}_{k=0}^{\infty}$  is *covariance stationary*, namely, its elements satisfy the relation  $\text{Cov}[\mathbf{X}_j^{(m)}, \mathbf{X}_{j+\tau}^{(n)}] = \text{Cov}[\mathbf{X}_k^{(m)}, \mathbf{X}_{k+\tau}^{(n)}]$ , that is, covariances depend on time only through the time lag  $\tau$  (see Section 2 for detailed explanation of the notation).

This paper proposes a solution methodology for the fitting problem above by introducing a new and versatile class of nonlinear multivariate stochastic sequences, called *MARM (Multivariate Auto-Regressive Modular)* processes. MARM processes are based on previous work designed to fit a stochastic model to a strong statistical signature of univariate empirical time series. Jagerman and Melamed (1992a, 1992b) proposed the class of *TES (Transform-Expand-Sample)* processes and the TES modeling methodology as an extension of basic TES processes [Melamed (1991)]. TES is a versatile class of stochastic sequences consisting of marginally uniform

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<sup>1</sup> A *statistical signature* is a set of realizable empirical or theoretic statistics, usually in model fitting context. Statistical signatures can be partially ordered under inclusion. For example, statistical signatures might contain first-order statistics (the common mean, variance and/or marginal distribution, etc.), second-order statistics (autocorrelation and/or cross-correlation functions etc.), or higher-order statistics.

autoregressive schemes with modulo-1 reduction, followed by various transformations. TES meets the three aforementioned goodness-of-fit criteria simultaneously in a systematic way. First, TES guarantees an exact fit to arbitrary marginal distributions (in particular, TES matches any empirical density). Secondly, TES affords considerable modeling flexibility in approximating empirical autocorrelations, including functional forms which are monotone, oscillatory, alternating and so on. Thirdly, TES processes span a wide qualitative range of sample paths, including cyclical as well as non-cyclical behavior. Altogether, TES is a nonlinear autoregressive scheme, yielding either Markovian or non-Markovian processes. TES processes further enjoy important computational advantages. TES sequences are easy to generate on a computer, and their generation time complexity is relatively small compared to that of the underlying pseudo-number generator, while their space complexity is negligible. Further, TES autocorrelations can be computed from accurate and fast (near real-time) analytical formulas without requiring simulation. A detailed discussion of TES processes and modeling methodology may be found in [Melamed (1993, 1997), Jagerman and Melamed (1992a, 1992b, 1994a, 1994b, 1995)]. A discretized variant of TES, called *QTES (Quantized TES)*, has been developed in [Melamed et al. (1996)] and used in [Klaoudatos et al. (1999)]. The class of *ARM (Auto-Regressive Modular)* processes generalizes the class of TES processes in the sense that ARM admits general so-called *innovation sequences* (see Section 3.1), whereas TES requires them to be *iid (independent identically distributed)*. Still, ARM processes, like TES processes, meet the three aforementioned goodness-of-fit criteria, even as ARM enhances modeling flexibility beyond TES.

This paper extends the class of ARM (univariate) processes to the class of MARM (multivariate) processes. Like ARM processes, MARM processes constitute a class of versatile stochastic processes, designed to fit a strong statistical signature of empirical vector-valued data. More specifically, while the ARM modeling methodology aims to capture a strong statistical signature of *univariate* empirical data by simultaneously fitting the empirical marginal distribution and leading empirical autocorrelations, the MARM modeling methodology aims to construct high-fidelity multivariate models from empirical *multivariate* time series, by simultaneously fitting first-order statistics (the multi-dimensional empirical histogram) and second-order statistics (empirical autocorrelations and cross-correlations). We mention that second-order statistics model some aspects of temporal dependence, which abounds in real-life phenomena, such as burstiness in telecommunications traffic [Fendick et al. (1989), Livny et al. (1993), Patuwo et al. (1993)], and clustering of extreme ocean climate events and evolution patterns in ecology [Aoki (1987)]. In particular, temporal dependence in traffic flows offered to queueing systems have motivated input analysis methods that can capture such dependence; see [Fendick et al. (1989), Livny et al. (1993)]. The autocorrelation and cross-correlation functions are convenient measures of linear temporal dependence in and across stochastic processes, respectively, and are frequently used by engineers as proxies for general temporal dependence [Bendat and Piersol (1986)].

Compared with the univariate ARM process, MARM processes are more complicated due to their dimensionality and the added temporal dependence across the components in addition to temporal dependence within components. Like ARM [Melamed (1999)], a MARM process consists of a *background process* and a *foreground process*, and a transformation that maps the former to the latter. Broadly speaking, the background process is in effect a hidden Markov chain [Rabiner (1989), Cappé et al. (2005)] whose state space has the *unit circle* topology (here, it means a circle with circumference 1), while the foreground process is obtained by a memoryless transformation. The background/foreground scheme ensures that the ARM/MARM model can be fitted exactly to any empirical histogram (in the case of MARM, to the multidimensional empirical histogram), while simultaneously approximating the leading empirical autocorrelations (and in the case of MARM also the leading cross-correlations).

The fitting procedures of ARM/MARM processes to empirical data are computationally hard, since the corresponding parameter state space is multidimensional [Brandt and Williams (2007)]. Fortunately, the transition structure of ARM/MARM processes can be derived in closed form and quickly evaluated by a computer through a search followed by gradient-driven optimization. The fitting procedure (called *CSLO* for *Comprehensive Search / Local Optimization*) generalizes the GSLO algorithm in [Jelenkovic and Melamed (1995a, 1995b)], and yields a set of candidate models, and the modeler can then decide which one to select. Finally, the transition structure is used to compute forecasts in the form of convex combinations of conditional expectations, and confidence intervals obtained from convex combinations of transition densities. For a detailed discussion of ARM processes and their modeling methodology, see [Melamed (1999)].

The rest of the paper is organized as follows. Section 2 provides some preliminaries including notation. Section 3 defines MARM processes and outlines some of their important properties. Section 4 and Section 5 study the MARM transition structure and autocorrelation and cross-correlation functions, respectively. Section 6 describes an algorithm for fitting MARM processes to a prescribed statistical signature, while Section 7 provides the fundamentals of the MARM forecasting methodology, including the forecasting statistics and choice of parameters. Finally, Section 8 concludes the paper.

A companion paper, Melamed and Zhao (2011), will specialize the general class of MARM processes to a practical sub-class suitable for modeling of empirical vector-valued time series, and will illustrate MARM modeling and forecasting using real-life data.

## 2. Preliminaries

In this section, we provide the necessary preliminaries, including notation and known results, to be used later in the construction of MARM processes.

Throughout the paper, we shall use the following notation. The bilateral Laplace transform of a function  $f(x)$  is defined by  $\tilde{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-sx} dx$ . The cumulative distribution function (cdf or distribution for short) and probability density function (pdf) of a random variable  $Z$  are denoted by  $F_Z$  and  $f_Z$  respectively, and the corresponding mean and standard deviation are denoted by  $\mu_Z$  and  $\sigma_Z$ .  $Z \sim F$  means that random variable  $Z$  has distribution  $F$ . In particular,  $\text{Unif}(I)$  denotes the uniform distribution on interval  $I$ . The indicator function of a set  $A$  is denoted by  $1_A(x)$ . The imaginary number  $\sqrt{-1}$  is denoted by  $i$ . For any vector  $\vec{v} = (v_1, \dots, v_n)$ , its sub-vector  $\vec{v}^{(k)}$ ,  $1 \leq k \leq n$ , is given by  $\vec{v}^{(k)} = (v_1, \dots, v_k)$ . The vector of integration is defined by  $d\vec{v}^{(k)} = dv_1 \dots dv_k$ . The juxtaposition operation applied to any two vectors,  $\vec{x} = (x_1, \dots, x_k)$  and  $\vec{y} = (y_1, \dots, y_n)$ , yields the vector  $(\vec{x}, \vec{y}) = (x_1, \dots, x_k, y_1, \dots, y_n)$ .

Let  $\{\mathbf{X}_k\}_{k=0}^{\infty}$  be a continuous-space real-valued (possibly multivariate) discrete-time stochastic process defined on a probability space  $(\Omega, \mathcal{F}, \Pr)$ , where  $\Omega$  is the sample space,  $\mathcal{F}$  is the sigma algebra, and  $\Pr$  is the probability measure. The realizations of  $\{\mathbf{X}_k\}_{k=0}^{\infty}$  are denoted by

$\{x_k\}_{k=0}^{\infty}$ , where  $x_k = X_k(\omega)$ ,  $\omega \in \Omega$ . In the multivariate case, the corresponding random sequence is denoted by  $\{\vec{X}_k\}_{k=0}^{\infty}$ , where  $\vec{X}_k = (X_k^{(1)}, X_k^{(2)}, \dots, X_k^{(N)})$ , and its realizations are denoted by  $\vec{x}_k = (x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(N)})$ , where  $x_k^{(j)} = X_k^{(j)}(\omega)$ ,  $\omega \in \Omega$ .

The autocorrelation function of a univariate stationary random sequence  $\{X_j^{(n)}\}_{j=0}^{\infty}$  with finite second moment is defined by

$$\rho_{n,n}(j, \tau) = \frac{\mathbf{E}[X_j^{(n)} X_{j+\tau}^{(n)}] - \mathbf{E}[X_j^{(n)}] \mathbf{E}[X_{j+\tau}^{(n)}]}{\sigma_{X^{(n)}}^2}, \quad (2.1)$$

and the cross-correlation function of stationary random sequences  $\{X_j^{(m)}\}_{j=0}^{\infty}$  and  $\{X_j^{(n)}\}_{j=0}^{\infty}$  is defined by

$$\rho_{m,n}(j, \tau) = \frac{\mathbf{E}[X_j^{(m)} X_{j+\tau}^{(n)}] - \mathbf{E}[X_j^{(m)}] \mathbf{E}[X_{j+\tau}^{(n)}]}{\sigma_{X^{(m)}} \sigma_{X^{(n)}}}. \quad (2.2)$$

When the correlation function depends only on the lag, we omit the first argument in Eqs. (2.1) and (2.2). The conditional density function of a random variable  $Y$  given a random variable  $X$  is denoted by

$$f_{Y|X}(y | x) = \frac{\partial}{\partial y} \Pr\{Y \leq y | X = x\}. \quad (2.3)$$

The *Inversion Method*, given in the next lemma, is a standard technique for transforming a uniform random variable to one with an arbitrary prescribed distribution  $F$ .

**Lemma 2.1 (Inversion Method)**

Let  $F$  be an arbitrary cumulative distribution function and  $U \sim \text{Unif}(0,1)$ , then the random variable  $X = F^{-1}(U)$  satisfies  $X \sim F$ . □

For a proof, see [Bratley et al. (1987), Law and Kelton (1991)]. The Inversion Method provides a very simple way of converting a marginally uniform sequence  $\{U_n\}$  into a sequence  $\{X_n\}$  with an arbitrary marginal distribution  $F$  by simply setting for each  $n$ ,

$$X_n = F^{-1}(U_n). \quad (2.4)$$

A transformation of the form (2.4) is called an *inversion transformation*. It always exists because  $F$ , being a cdf, is non-decreasing, and therefore can always be inverted; although the inversion may not be unique (unless  $F$  is strictly increasing), all choices of inversions produce the same effect when applying the Inversion Method.

A *step-function inversion* is a special case of the Inversion Method when the underlying density is a step function of the form

$$f(y) = \sum_{i=1}^I 1_{[l_i, r_i)}(y) \frac{p_i}{w_i}, \quad -\infty < y < \infty, \quad (2.5)$$

where  $I$  is the number of the steps, each of the form  $[l_i, r_i)$ ,  $w_i = r_i - l_i > 0$  is the width of step  $i$ , and  $p_i$  is the probability of step  $i$ . The corresponding cdf is the piecewise linear function

$$F(\mathbf{y}) = \sum_{j=1}^J 1_{[l_j, r_j)}(\mathbf{y}) \left[ C_{j-1} + (\mathbf{y} - l_j) \frac{p_j}{w_j} \right], \quad -\infty < \mathbf{y} < \infty, \quad (2.6)$$

where  $\{C_j\}$  is the cdf of the probability density  $\{p_j\}_{j=1}^J$ , i.e.,  $C_j = \sum_{i=1}^j p_i$ ,  $1 \leq j \leq J$ , with  $C_0 = 0$  and  $C_J = 1$ . The inverse function of  $F$ , is the piecewise-linear function

$$F^{-1}(\mathbf{x}) = \sum_{j=1}^J 1_{[C_{j-1}, C_j)}(\mathbf{x}) \left[ l_j + (\mathbf{x} - C_{j-1}) \frac{w_j}{p_j} \right], \quad 0 \leq \mathbf{x} \leq 1. \quad (2.7)$$

Eq. (2.7) is just a special case of the Inversion Method for piecewise-linear inversions. The MARM modeling methodology uses inversions of the form (2.7) to construct candidate models with prescribed step-function densities (typically estimated from empirical data).

In practice, modelers often estimate a step-function density of an empirical sample via an empirical histogram. That is, for an empirical sequence  $\vec{\mathbf{y}} = \{\mathbf{y}_j\}_{j=0}^{J-1}$ , the corresponding empirical density  $f(\mathbf{y})$  in Eq. (2.5) is often estimated by a *histogram* pdf of the form

$$\hat{h}_{\vec{\mathbf{y}}}(\mathbf{y}) = \sum_{i=1}^I 1_{[l_i, r_i)}(\mathbf{y}) \frac{\hat{p}_i}{w_i}, \quad -\infty < \mathbf{y} < \infty, \quad (2.8)$$

where  $\hat{p}_i$  is obtained from some histogram of the empirical sequence as the probability estimate of cell  $i$  (relative frequency of observations that fell into that cell). Here and elsewhere, hats indicate sample-based estimated quantities, as opposed to modeler-supplied parameters. Accordingly, the corresponding *histogram cdf*,  $\hat{H}_{\vec{\mathbf{y}}}$ , and *histogram inverse*  $\hat{H}_{\vec{\mathbf{y}}}^{-1}$  are obtained from Eqs. (2.6) and (2.7), respectively, by substituting  $p_j$  with  $\hat{p}_j$ .

The merit of the histogram inversion approach in practical modeling is its broad generality, since no assumptions are made on the actual form of the true marginal density. However, to be a good approximation of the true marginal density, sufficient data must be used.

The MARM modeling methodology makes use of the Inversion Method to construct multivariate random variables, and to this end, one applies it to conditional distributions. We next illustrate this construction for the  $N$ -dimensional case. Let  $\vec{\mathbf{X}} = (\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(N)})$  be a random vector of dimension  $N$ , with the standard representation for its joint density,

$$f_{\vec{\mathbf{X}}}(\vec{\mathbf{x}}) = f_{\mathbf{X}^{(1)}}(\mathbf{x}) f_{\mathbf{X}^{(2)}|\mathbf{X}^{(1)}}(\mathbf{x}^{(2)} | \mathbf{x}^{(1)}) \cdots f_{\mathbf{X}^{(N)}|\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(N-1)}}(\mathbf{x}^{(N)} | \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N-1)}), \quad (2.9)$$

where  $\vec{\mathbf{x}} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)})$ .

### **Algorithm 2.1: Random Vector Generating Algorithm (RVGA)**

**Input:** A joint pdf  $f_{\vec{X}}(\vec{x})$ , and a random vector  $\vec{W} = (W^{(1)}, \dots, W^{(N)})$  of iid Unif[0,1) components,  $W^{(k)}$ ,  $k = 1, \dots, N$ . From the pdf, we compute all marginal and conditional cdf, which are also assumed given.

**Output:** A random vector  $\vec{X} = (X^{(1)}, \dots, X^{(N)})$  with the prescribed joint pdf  $f_{\vec{X}}(\vec{x})$ .

**Algorithm:**

1. For  $k = 1$ , define

$$X^{(1)} = F_{X^{(1)}}^{-1}(W^{(1)}) \quad (2.10)$$

2. For  $k > 1$ , the definition of  $X^{(k)}$  proceeds recursively. Assume that  $X^{(1)}, \dots, X^{(k-1)}$  have already been defined, and define

$$X^{(k)} = F_{X^{(k)}|X^{(1)}, \dots, X^{(k-1)}}^{-1}(W^{(k)}). \quad (2.11)$$

### **Lemma 2.2**

The random vector  $\vec{X} = (X^{(1)}, \dots, X^{(N)})$  defined by RVGA has the joint pdf  $f_{\vec{X}}(\vec{x})$ .

**Proof:**

By construction,  $X^{(1)} \sim F_{X^{(1)}}$  as a consequence of the Inversion Method in Lemma 2.1. For  $1 < k \leq N$ , we similarly have  $X^{(k)} \sim F_{X^{(k)}|\vec{X}^{(k-1)}}$  by virtue of the Inversion Method applied to conditional distributions. Thus, the joint pdf of  $\vec{X} = (X^{(1)}, \dots, X^{(N)})$  is given by (2.9), as required.  $\square$

We point out that from a simulation viewpoint, RVGA provides the basis for computer generation of realizations  $\vec{x} = (x^{(1)}, \dots, x^{(N)})$  from the joint pdf  $f_{\vec{X}}(\vec{x})$ , via replacing all random variables in RVGA by their realizations.

## **3. Construction of MARM Processes**

A MARM process is constructed using two components: a background process which is an auxiliary hidden Markov chain, and a foreground process which is our target time series sequence. We shall refer to the aforementioned construction as foreground/background scheme. The main advantage of the background/foreground paradigm to be presented is its broad modeling flexibility, as will be later demonstrated in this paper.

In this section, we describe the construction of two flavors of MARM processes,  $\text{MARM}^+$  and  $\text{MARM}^-$ . For notational clarity, we will append the plus or minus symbols to objects associated with the respective MARM process, and omit those symbols when they are common to both flavors of MARM processes.

### 3.1. MARM Background Processes

The MARM modeling methodology uses *Modulo-1 arithmetic*, which is the operation of taking the fractional part of a real number. Letting  $\lfloor x \rfloor = \max\{\text{integer } n : n \leq x\}$  be the *floor operator*, the *modulo-1 (fractional part) operator*  $\langle \cdot \rangle$  is defined for any real  $x$  by  $\langle x \rangle = x - \lfloor x \rfloor$ . Note that the fractional part always lies in the interval  $[0, 1)$ , even for negative numbers. For example,  $\langle 1.1 \rangle = 0.1$ , but  $\langle -1.1 \rangle = 0.9$ . Other useful properties of modulo-1 addition are

$$\langle a \pm b \rangle = \langle \langle a \rangle \pm \langle b \rangle \rangle \quad (3.1)$$

and

$$\langle a + b \rangle = \langle c \rangle \Leftrightarrow \langle a \rangle = \langle c - b \rangle. \quad (3.2)$$

A geometric representation of modulo-1 arithmetic can be found in [22], where the interval  $[0, 1)$  (the range of fractional parts) has been topologically transformed into a circle in a distance-preserving manner. This circle is referred to as the *unit circle* (circumference is unity, not the radius), and its origin (the fraction 0) is arbitrarily selected, say, on the bottom of the unit circle. A number  $0 \leq y < 1$  is represented by a point on the unit circle at distance  $y$  from the origin (the distance is measured by the length of the arc connecting the origin clockwise to  $y$ ). For any real  $y$  (not necessary in  $[0, 1)$ ),  $y$  is represented on the unit circle with the same position as its fractional part  $\langle y \rangle$ . For a geometric description of modulo-1 arithmetic, see [Melamed (1993)].

We next proceed to construct the MARM background process. Hereafter, all random variables will be defined over some common probability space. Let  $U_0 \sim \text{Unif}[0, 1)$ . An *innovation sequence* is any random sequence,  $\{V_n\}_{n=0}^{\infty}$ , independent of  $U_0$ . An important class of innovations consists of iid sequences. Let  $S_{m,n}^{(V)}$  be the partial sum of innovation random variables, given by

$$S_{m,n}^{(V)} = \begin{cases} 0, & \text{if } m > n \\ \sum_{j=m}^n V_j, & \text{if } m \leq n \end{cases} \quad (3.3)$$

#### 3.1.1 MARM<sup>+</sup> Background Processes

A MARM<sup>+</sup> background sequence,  $\{U_n^+\}_{n=0}^{\infty}$ , is defined recursively by

$$U_n^+ = \begin{cases} U_0, & n = 0 \\ \langle U_{n-1}^+ + V_n \rangle, & n > 0 \end{cases} \quad (3.4)$$

The construction of  $\{U_n^+\}_{n=0}^{\infty}$  has a simple and intuitive geometric interpretation: the modulo-1 arithmetic gives rise to Markovian walks on the unit circle, where each point on the circle corresponds to a fraction in the range  $[0, 1)$  [Melamed (1993)].

Using Eq. (3.1), we can equivalently represent  $U_n^+$  in (3.4) as



$$U_n^+ = \left\langle U_0 + S_{0,n}^{(V)} \right\rangle = \left\langle U_0 + \left\langle S_{0,n}^{(V)} \right\rangle \right\rangle, \quad n \geq 0 \quad (3.5)$$

The motivation for defining MARM background processes stems from the simple fact that uniformity is closed under modulu-1 addition of independent random variable. This fact is due to the following lemma, called General Iterated Uniformity in [22]Melamed (1993).

**Lemma 3.1 (General Iterated Uniformity)**

Let  $U \sim \text{Unif}[0, 1)$  and  $X = \langle U + V \rangle$ , where  $V$  is an arbitrarily distributed random variable, independent of  $U$ . Then,  $X \sim \text{Unif}[0, 1)$ . □

For a proof, see [Melamed (1993, 1999)]. An important consequence of this lemma and Eq. (3.4) is given in the following corollary.

**Corollary 3.1**

MARM<sup>+</sup> background sequences are Markovian, and their marginal distributions are stationary and uniform on  $[0, 1)$  regardless of the probability law of the innovation sequences  $\{V_n\}$  selected. □

For a proof, see [Melamed (1999)].

**3.1.2 MARM<sup>-</sup> Background Processes**

A MARM<sup>-</sup> background sequence,  $\{U_n^-\}_{n=0}^\infty$ , is defined by

$$U_n^- = \begin{cases} U_n^+, & n \text{ even} \\ 1 - U_n^+, & n \text{ odd} \end{cases} \quad (3.6)$$

**Corollary 3.2**

MARM<sup>-</sup> background sequences are non-stationary Markovian, and their marginal distributions are uniform on  $[0, 1)$  regardless of the probability law of the innovation sequences  $\{V_n\}$  selected.

**Proof.**

Follows from Eq. (3.6) and Corollary 3.1. □

**3.2. MARM Stitched Background Processes**

A random walk on a unit circle (details to be discussed in section 4.1) usually produces visual “discontinuities” when crossing the circle origin in a clockwise or counter-clockwise direction. Here the term “discontinuities” is used figuratively, referring to transitions from large fractions to small ones and vice versa, when the random walk crosses with the origin via the modulo-1 operation in each direction.

In some scenarios, such sample “discontinuities” could have a valid modeling interpretation. For example, crossing point 0 could model a catastrophe when properly defined, such as in a cyclical economic model. However, in most cases, such “discontinuities” in the sample path are undesirable, since they have no counterparts in empirical data. Monotone transformations (including the frequently used Inversion Method) preserve such “discontinuities”. To satisfy the goodness-of-fit criteria discussed in Section 1, some uniformity-preserving “smoothing” transformations need to be applied in order to “remove” such “discontinuities” while still allowing us to fit arbitrary empirical marginal distributions via the Inversion Method. To this end,

one may use the so-called *stitching transformations* [Jagerman and Melamed (1992a), Melamed (1993)].

Formally, a stitching transformation  $S_\xi$  maps the interval  $[0,1]$  to itself, and is determined by a *stitching parameter*  $\xi$  in the range  $[0,1]$ . For a given  $\xi$ ,  $S_\xi$  is defined by

$$S_\xi(u) = \begin{cases} u / \xi, & 0 \leq u \leq \xi \\ (1 - u) / (1 - \xi), & \xi \leq u \leq 1 \end{cases} \quad (3.7)$$

The term “stitching” is motivated by the fact that  $S_\xi$  is continuous on the unit circle as function of  $\xi$  in  $(0,1)$ . Note that for the trivial cases ( $\xi = 0$  or  $1$ ), we have  $S_0(u) = 1 - u$  (the antithetic transformation) and  $S_1(u) = u$  (the identity transformation), where no “smoothing” takes place since they are not continuous at the edge points 0 and 1 on the unit circle.

The primary usage of stitching transformations is their uniformity-preserving property, in addition to their “smoothing” effect, which is guaranteed by the following lemma.

**Lemma 3.2**

If  $U \sim \text{Unif}[0, 1)$ , then  $S_\xi(U) \sim \text{Unif}[0, 1)$ , for all  $0 \leq \xi \leq 1$ . □

For a proof, see [21]. It follows from Lemma 2.1 and Lemma 3.2 that a composition mapping (called a *distortion*) of the form

$$D(u) = F_X^{-1}(S_\xi(u)), \quad u \in [0,1), \quad (3.8)$$

employing stitching and inversion in succession on a random variable uniformly distributed in  $[0,1)$ , will allow us to fit arbitrary empirical marginal distributions, and simultaneously provide sample path “smoothing” for  $\xi$  in  $(0,1)$ . The MARM modeling methodology, to be described in the next section, utilizes distortions of the form (3.8).

Lemma 3.1 ensures that a MARM background sequence  $\{U_n\}_{n=0}^\infty$  is a sequence of random variables, uniformly distributed in  $[0,1)$ , which constitute a Markovian Walk on the unit circle. From the discussion above, such a process usually has visual discontinuities in its sample paths stemming from the underlying modulo-1 arithmetic. As per the forgoing discussion, stitching transformations will be needed to “smooth” MARM sample paths of the background process, while preserving the uniformity of the marginal distribution. The MARM modeling methodology utilizes stitched MARM background processes, namely  $\{S_\xi(U_n)\}_{n=0}^\infty$  (where  $\xi \in [0,1]$  is the stitching parameter) to generate MARM foreground processes, which will be discussed in detail in the next section.

**3.3. MARM Foreground Processes**

A *MARM foreground process* is an  $N$ -dimensional stochastic process  $\{\vec{X}_n\}_{n=0}^\infty$  with some state space  $\mathcal{S} = \mathcal{S}^{(1)} \times \dots \times \mathcal{S}^{(N)} \subset \mathbb{R}^N$ . It is constructed from a MARM background process via a memoryless transformation, called a *distortion* [cf. Melamed (1993, 1999)]. There are many kinds of distortions that can be applied to the background processes, which partly gives the reason for the versatile modeling flexibility of MARM modeling methodology. In this section, we are

interested in two special distortions based on the stitching transformations and the Inversion Method.

Generally speaking, the construction of the MARM foreground sequence,  $\{\vec{X}_j\}_{j \geq 0}$ , follows the random vector generating algorithm (RVGA) described in Section 2. RVGA generates the corresponding  $\vec{X}_j$ ,  $j \geq 0$ , through distortions including stitching transformations and Inversion Method. We shall be concerned with distortion functions  $D^{(k)}(\mathbf{u}, \vec{w}^{(k-1)})$  and vector valued distortions of the form  $\vec{D}^{(k)} = (D^{(1)}, \dots, D^{(k)})$ .

The construction is specified by the following algorithm.

**Algorithm 3.1: MARM Foreground Process Construction Algorithm**

**Input:** A MARM background sequence  $\{U_j\}_{j=0}^\infty$  (either  $\{U_j^+\}_{j=0}^\infty$  or  $\{U_j^-\}_{j=0}^\infty$ ); a stitching parameter  $\xi$ ; iid random variables  $W_j^{(k)} \sim \text{Unif}[0,1)$ ,  $1 \leq k \leq N$ ,  $j \geq 0$ , which are all independent of  $\{U_j\}_{j=0}^\infty$ ; a joint pdf,  $f_{\vec{X}}(\vec{x})$ , from which we compute all marginal and conditional cdfs, as well as their inverses to derive the distortions of the form.

**Output:** A vector of distortions,  $\vec{D}^{(k)} = (D^{(1)}, \dots, D^{(k)})$ , and a MARM foreground processes  $\{\vec{X}_j\}_{j=0}^\infty$  with the prescribed joint pdf  $f_{\vec{X}}(\vec{x})$ .

**Algorithm:** For each  $j \geq 0$ , apply the following steps:

**Step 1:** For  $k = 1$ , apply the stitching transformation to  $U_j$ , yielding the corresponding stitched MARM background random variable,  $S_\xi(U_j)$ . Define  $D^{(1)}$  and  $\{X_j^{(1)}\}$  by

$$D^{(1)}(\mathbf{u}) = F_{X^{(1)}}^{-1}(S_\xi(\mathbf{u})), \mathbf{u} \in [0,1) \quad (3.9)$$

$$X_j^{(1)} = D^{(1)}(U_j), j \geq 0. \quad (3.10)$$

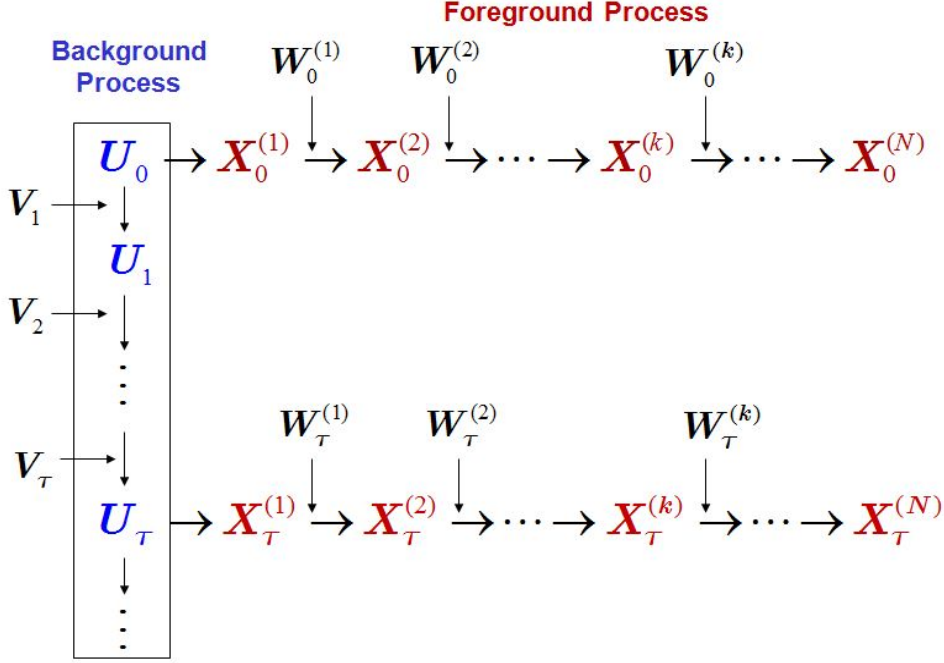
**Step 2:** For  $1 < k \leq N$ ,  $D^{(k)}$  and  $\{X_j^{(k)}\}$  are defined recursively. Assuming that  $\vec{X}_j^{(k-1)} = (X_j^{(1)}, \dots, X_j^{(k-1)})$  has already been defined, let  $D^{(k)}$  and  $\{X_j^{(k)}\}$  be defined by

$$D^{(k)}(\mathbf{w}, \vec{x}^{(k-1)}) = F_{X^{(k)} | \vec{X}^{(k-1)}}^{-1}(\mathbf{w} | \vec{x}^{(k-1)}), \mathbf{w} \in [0,1), \vec{x}^{(k-1)} \in S^{(1)} \times \dots \times S^{(k-1)} \quad (3.11)$$

$$X_j^{(k)} = D^{(k)}(W_j^{(k-1)}, \vec{X}_j^{(k-1)}), j \geq 0 \quad (3.12)$$

□

The background/foreground structure of MARM processes is shown schematically in Figure 3.1.



**Figure 3.1** Background/Foreground scheme for constructing MARM processes

In Figure 3.1, arrows denote two kinds of “transitions: horizontal transitions correspond to transformations from  $\vec{X}_j^{(n)}$  to  $\vec{X}_j^{(n+1)}$  for  $j \geq 0$  and  $n \geq 1$ , while vertical ones correspond to background process transitions from  $U_i$  to  $U_{i+1}$  for any  $i \geq 0$ .

### 3.4. Computer Generation of MARM Processes

Computer generation of MARM foreground processes  $\{\vec{X}_j\} = \{X_j^{(k)}\}$ ,  $1 \leq k \leq N$ ,  $j \geq 0$ , follows Algorithm 3.1, except that random variables are replaced by their realizations. Lemma 2.2, Lemma 3.1 and Lemma 3.2 then jointly ensure that the resultant MARM foreground processes have the prescribed marginal distribution. To complete the generation procedures, we need to generate realizations of  $U_0$ , the innovation variates,  $\{V_n\}$ , and the auxiliary variates  $\{W_j^{(k)}\}$ . All these are obtained, using some random number generator, namely, a computer-generated random sequence  $\{Z_n\}_{n \geq 0}$ , which is iid  $\text{Unif}[0,1)$ . For example, for  $1 \leq k \leq N$  and  $j \geq 0$ , the random number stream,  $\{Z_n\}_{n \geq 1}$ , may be used as follows:

1.  $U_0 \leftarrow Z_0$
2.  $V_j \leftarrow F_V^{-1}(Z_{jN})$
3.  $W_j^{(k)} \leftarrow Z_{jN+k}$

## 4. The Transition Structure of MARM Processes

In this section, we will derive the transition structure of MARM processes, which will later be used to compute the autocorrelations/cross-correlations. For any MARM background process  $\{U_n\}$  and  $\tau \geq 1$ , we shall use in the sequel the relationship

$$f_{U_j|U_{j+\tau}}(u_j | u_{j+\tau}) = f_{U_{j+\tau}|U_j}(u_{j+\tau} | u_j) \frac{f_{U_j}(u_j)}{f_{U_{j+\tau}}(u_{j+\tau})} = f_{U_{j+\tau}|U_j}(u_{j+\tau} | u_j) \quad (4.1)$$

which readily follows from the fact that the marginal density of a MARM background process is uniform over  $[0,1)$ .

### Lemma 4.1

For any function  $g(x)$  and integer  $\nu$ ,  $\tilde{g}(i2\pi\nu)$  and  $\tilde{g}(-i2\pi\nu)$  are complex conjugate pairs.

#### Proof.

Follows from the representations

$$\begin{aligned} \tilde{g}(i2\pi\nu) &= \int g(x) e^{-i2\pi\nu x} dx = \int g(x) [\cos(2\pi\nu x) - i \sin(2\pi\nu x)] dx \\ \tilde{g}(-i2\pi\nu) &= \int g(x) e^{i2\pi\nu x} dx = \int g(x) [\cos(2\pi\nu x) + i \sin(2\pi\nu x)] dx. \quad \square \end{aligned}$$

### Theorem 4.1

Let  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $\mathbf{x}^{(k)} \in \mathcal{S}^{(k)}$ ,  $1 \leq k \leq N$ .

(a) For  $\tau = 0$  and  $1 \leq k \leq N$ ,

$$f_{\mathbf{X}_j^{(k)}|U_j}(\mathbf{x}^{(k)} | u_j) = \begin{cases} 1_{\{F_{\mathbf{X}^{(1)}}^{-1}(\mathcal{S}_\xi(u_j))\}}(\mathbf{x}^{(1)}), & k = 1 \\ \int_{\mathcal{S}^{(2)}} \dots \int_{\mathcal{S}^{(k-1)}} \frac{f_{\tilde{\mathbf{X}}^{(k)}}(F_{\mathbf{X}^{(1)}}^{-1}(\mathcal{S}_\xi(u_j)), \mathbf{y}^{(2)}, \dots, \mathbf{y}^{(k-1)}, \mathbf{x}^{(k)})}{f_{\mathbf{X}^{(1)}}(F_{\mathbf{X}^{(1)}}^{-1}(\mathcal{S}_\xi(u_j)))} d\mathbf{y}^{(2)} \dots d\mathbf{y}^{(k-1)}, & k > 1 \end{cases} \quad (4.2)$$

(b) For  $\tau \geq 0$  and  $k = 1$ ,

$$\begin{aligned} f_{\mathbf{X}_{j+\tau}^{(1)}|U_j}(\mathbf{x}^{(1)} | u_j) &= \\ & f_{U_{j+\tau}|U_j}(\xi F_{\mathbf{X}^{(1)}}(\mathbf{x}^{(1)}) | u_j) \xi f_{\mathbf{X}^{(1)}}(\mathbf{x}^{(1)}) + \\ & f_{U_{j+\tau}|U_j}(1 - (1 - \xi) F_{\mathbf{X}^{(1)}}(\mathbf{x}^{(1)}) | u_j) (1 - \xi) f_{\mathbf{X}^{(1)}}(\mathbf{x}^{(1)}) \end{aligned} \quad (4.3)$$

#### Proof.

The first line of part (a) follows immediately since  $\mathbf{X}_j^{(1)} = F_{\mathbf{X}^{(1)}}^{-1}(\mathcal{S}_\xi(U_j))$  by construction of MARM foreground processes in Eq. (3.10).

To prove the second line of part (a), note that by construction of MARM processes for  $k > 1$ ,

$$f_{X_j^{(2)}, \dots, X_j^{(k)} | X_j^{(1)}, U_j} = f_{X^{(2)}, \dots, X^{(k)} | X^{(1)}}.$$

Consequently, using this fact and the first line of part (a), we have

$$\begin{aligned} f_{\bar{X}_j^{(k)} | U_j}(\bar{\mathbf{x}}^{(k)} | \mathbf{u}_j) &= f_{X_j^{(1)} | U_j}(\mathbf{x}^{(1)} | \mathbf{u}_j) f_{X_j^{(2)}, \dots, X_j^{(k)} | X_j^{(1)}, U_j}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)} | \mathbf{x}^{(1)}, \mathbf{u}_j) \\ &= f_{X_j^{(1)} | U_j}(\mathbf{x}^{(1)} | \mathbf{u}_j) f_{X^{(2)}, \dots, X^{(k)} | X^{(1)}}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)} | \mathbf{x}^{(1)}) \\ &= 1_{\{\mathbf{x}^{(1)} = F_{X^{(1)}}^{-1}(\mathcal{S}_\xi(\mathbf{u}_j))\}}(\mathbf{x}^{(1)}) \frac{f_{\bar{X}^{(k)}}(\bar{\mathbf{x}}^{(k)})}{f_{X^{(1)}}(\mathbf{x}^{(1)})} \end{aligned}$$

Integrating the above equation with respect to  $\bar{\mathbf{x}}^{(k-1)}$  now completes the proof of Eq. (4.2).

To prove part (b), write

$$\begin{aligned} F_{X_{j+\tau}^{(1)} | U_j}(\mathbf{x}^{(1)} | \mathbf{u}_j) &= \Pr\{X_{j+\tau}^{(1)} \leq \mathbf{x}^{(1)} | U_j = \mathbf{u}_j\} \\ &= \Pr\{F_{X^{(1)}}^{-1}(\mathcal{S}_\xi(U_{j+\tau})) \leq \mathbf{x}^{(1)} | U_j = \mathbf{u}_j\} \\ &= \Pr\{\mathcal{S}_\xi(U_{j+\tau}) \leq F_{X^{(1)}}(\mathbf{x}^{(1)}) | U_j = \mathbf{u}_j\} \\ &= \Pr\{U_{j+\tau} \leq \xi F_{X^{(1)}}(\mathbf{x}^{(1)}) | U_j = \mathbf{u}_j\} + \\ &\quad \Pr\{U_{j+\tau} \geq 1 - (1 - \xi) F_{X^{(1)}}(\mathbf{x}^{(1)}) | U_j = \mathbf{u}_j\} \\ &= F_{U_{j+\tau} | U_j}(\xi F_{X^{(1)}}(\mathbf{x}^{(1)}) | \mathbf{u}_j) + 1 - F_{U_{j+\tau} | U_j}(1 - (1 - \xi) F_{X^{(1)}}(\mathbf{x}^{(1)}) | \mathbf{u}_j) \end{aligned} \tag{4.4}$$

Differentiating Eq. (4.4) with respect to  $\mathbf{x}^{(1)}$  thus yields Eq. (4.3).  $\square$

#### Corollary 4.1

Let  $0 \leq \mathbf{u}_j < 1$ ,  $j \geq 0$ , and  $\mathbf{x}^{(k)} \in \mathcal{S}^{(k)}$ ,  $1 \leq k \leq N$ .

(a) For  $\tau = 0$  and  $1 \leq k \leq N$ ,

$$F_{X_j^{(k)} | U_j}(\mathbf{x}^{(k)} | \mathbf{u}_j) = \begin{cases} 1, & k = 1 \text{ and } \mathbf{x}^{(1)} \geq F_{X^{(1)}}^{-1}(\mathcal{S}_\xi(\mathbf{u}_j)) \\ 0, & k = 1 \text{ and } \mathbf{x}^{(1)} < F_{X^{(1)}}^{-1}(\mathcal{S}_\xi(\mathbf{u}_j)) \\ \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{\mathcal{S}^{(k-1)}} \dots \int_{\mathcal{S}^{(2)}} \frac{f_{\bar{X}^{(k)}}(F_{X^{(1)}}^{-1}(\mathcal{S}_\xi(\mathbf{u}_j)), \mathbf{y}^{(2)}, \dots, \mathbf{y}^{(k)})}{f_{X^{(1)}}(F_{X^{(1)}}^{-1}(\mathcal{S}_\xi(\mathbf{u}_j)))} d\mathbf{y}^{(2)} \dots d\mathbf{y}^{(k-1)} d\mathbf{y}^{(k)}, & k > 1 \end{cases} \tag{4.5}$$

(b) For  $\tau \geq 0$  and  $k = 1$ ,

$$F_{X_{j+\tau}^{(1)} | U_j}(\mathbf{x}^{(1)} | \mathbf{u}_j) = F_{U_{j+\tau} | U_j}(\xi F_{X^{(1)}}(\mathbf{x}^{(1)}) | \mathbf{u}_j) + 1 - F_{U_{j+\tau} | U_j}(1 - (1 - \xi) F_{X^{(1)}}(\mathbf{x}^{(1)}) | \mathbf{u}_j) \tag{4.6}$$

**Proof.**

The first and second lines of Part (a) are immediate from the first line of Eq.(4.2), while the third line of part (a) follows by integrating the corresponding densities in part (a) of Theorem 4.1. Finally, part (b) has already been proven in part (b) of Theorem 4.1.  $\square$

In the sequel, we shall employ the following notation. An alternative representation of the vector distortion operator  $\vec{D}^{(k)} = (D^{(1)}, \dots, D^{(k)})$  from Algorithm 3.1 is

$$D^{(1)}(\mathbf{u}, \mathbf{w}) = D^{(1)}(\mathbf{u}) = F_{X^{(1)}}^{-1}(S_\xi(\mathbf{u})) \quad (4.7)$$

$$D^{(k)}(\mathbf{u}, \vec{\mathbf{w}}^{(k-1)}) = F_{X^{(k)}|\vec{X}^{(k-1)}}^{-1}(\mathbf{w}^{(k-1)} | \vec{D}^{(k-1)}(\mathbf{u}, \vec{\mathbf{w}}^{(k-2)})), \quad 1 < k \leq N \quad (4.8)$$

Finally, the family of functions  $\mathcal{D}^{(k)}(\mathbf{u})$  is given by

$$\mathcal{D}^{(k)}(\mathbf{u}) = \begin{cases} D^{(1)}(\mathbf{u}), & k = 1 \\ \int_0^1 \dots \int_0^1 D^{(k)}(\mathbf{u}, \vec{\mathbf{w}}^{(k-1)}) d\vec{\mathbf{w}}^{(k-1)}, & 1 < k \leq N \end{cases} \quad (4.9)$$

where  $D^{(1)}(\mathbf{u})$  and  $D^{(k)}(\mathbf{u}, \vec{\mathbf{w}}^{(k-1)})$  are the distortion functions defined in Eqs. (4.7) and (4.8), respectively.

**Corollary 4.2**

For  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $1 \leq k \leq N$ ,

$$\mathbb{E}[X_j^{(k)} | U_j = u_j] = \mathcal{D}^{(k)}(u_j) \quad (4.10)$$

**Proof.**

Follows from Algorithm 3.1 and representation in Eq. (4.9).  $\square$

**Lemma 4.2**

For  $1 \leq k \leq N$ ,

$$\tilde{\mathcal{D}}^{(k)}(0) = \mu_{X^{(k)}}. \quad (4.11)$$

**Proof.**

$$\begin{aligned} \tilde{\mathcal{D}}^{(k)}(0) &= \int_0^1 \mathcal{D}^{(k)}(\mathbf{u}) d\mathbf{u} \\ &= \int_0^1 \dots \int_0^1 D^{(k)}(\mathbf{u}, \mathbf{w}^{(1)}, \dots, \mathbf{w}^{(k-1)}) d\mathbf{u} d\mathbf{w}^{(1)} \dots d\mathbf{w}^{(k-1)} \\ &= \int_{S^{(1)}} \dots \int_{S^{(k)}} \mathbf{x}^{(k)} f_{X^{(1)}}(\mathbf{x}^{(1)}) f_{X^{(2)}|\vec{X}^{(1)}}(\mathbf{x}^{(2)} | \mathbf{x}^{(1)}) \dots f_{X^{(k)}|\vec{X}^{(k-1)}}(\mathbf{x}^{(k)} | \vec{\mathbf{x}}^{(k-1)}) d\vec{\mathbf{x}}^{(k)} \quad (4.12) \\ &= \int_{S^{(1)}} \dots \int_{S^{(k)}} \mathbf{x}^{(k)} f_{X^{(k)}|\vec{X}^{(k-1)}}(\mathbf{x}^{(k)} | \vec{\mathbf{x}}^{(k-1)}) f_{\vec{X}^{(k-1)}}(\vec{\mathbf{x}}^{(k-1)}) d\vec{\mathbf{x}}^{(k)} \\ &= \mu_{X^{(k)}} \end{aligned}$$

where the second equality is given by Eq. (4.9).  $\square$

We point out that since the case  $\tau = 0$  has already been treated above, we will only treat the case  $\tau > 0$  in the rest of this section.

### 4.1. Transition Structure of MARM<sup>+</sup> Processes

The following theorem is a fundamental representation for the transition density function,  $f_{U_{j+\tau}^+|U_j^+}(v|u)$ , of MARM<sup>+</sup> background processes.

#### Theorem 4.2

$\{U_n^+\}$  is Markovian with stationary transition density  $f_{U_{j+\tau}^+|U_j^+}(v|u)$ , where for  $\tau > 0$  and  $0 \leq u, v < 1$ ,

$$f_{U_{j+\tau}^+|U_j^+}(v|u) = \begin{cases} 1_{\{u\}}(v), & \tau = 0, 0 \leq v, u < 1 \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu(v-u)} \right], & \tau > 0, 0 \leq v, u < 1 \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu(u-v)} \right], & \tau < 0, 0 \leq v, u < 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.13)$$

#### Proof.

We first prove the representation

$$f_{U_{j+\tau}^+|U_j^+}(v|u) = \begin{cases} 1_{\{u\}}(v), & \tau = 0, 0 \leq v, u < 1 \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu(v-u)}, & \tau > 0, 0 \leq v, u < 1 \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu(u-v)}, & \tau < 0, 0 \leq v, u < 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.14)$$

For  $\tau = 0$ , the first line of Eq. (4.14) is trivial. For  $\tau > 0$  and  $0 \leq v, u < 1$ , the proof appears in Theorem 1 in [Melamed (1999)]. For  $\tau < 0$  and  $0 \leq v, u < 1$ , it follows from Eq. (4.1) and the case for positive  $\tau$  above. Eq. (4.13) now follows from Eq. (4.14) with the aid of Lemma 4.1 and the fact that  $\tilde{f}_{S_{j+1, j+\tau}}(0) = \tilde{f}_{S_{j+\tau+1, j}}(0) = 1$ .  $\square$

We next proceed to exhibit the transition structure for the foreground process  $\{\bar{X}_n^+\}$ .

#### Theorem 4.3

Let  $\tau > 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $\bar{x}^{(k)} \in S^{(1)} \times \dots \times S^{(k)}$ ,  $x^{(k)} \in S^{(k)}$ ,  $k \geq 1$ .

$$(a) f_{\bar{X}_{j+\tau}^{(k)+}|U_j^+}(\bar{x}^{(k)}|u_j) = f_{\bar{X}^{(k)+}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \left( \xi e^{i2\pi\nu \xi F_{X^{(1)+}}(x^{(1)})} + (1 - \xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)+}}(x^{(1)})} \right) \right] \right\} \quad (4.15)$$



$$\begin{aligned}
\text{(b) } f_{\bar{X}_{j-\tau}^{(k)+} | U_j^+}(\bar{\mathbf{x}}^{(k)} | \mathbf{u}_j) &= \\
f_{\bar{X}^{(k)+}(\bar{\mathbf{x}}^{(k)})} &\left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
&\left. \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)+}(\mathbf{x}^{(1)})}} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)+}(\mathbf{x}^{(1)})}} \right) \right] \right\} \quad (4.16)
\end{aligned}$$

$$\begin{aligned}
\text{(c) } f_{X_{j+\tau}^{(k)+} | U_j^+}(\mathbf{x}^{(k)} | \mathbf{u}_j) &= \\
f_{X^{(k)+}(\mathbf{x}^{(k)})} &+ 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1}, j+\tau} (i2\pi\nu) e^{-i2\pi\nu u_j} \int_{S^{(1)}} \dots \int_{S^{(k-1)}} f_{\bar{X}^{(k)+}(\bar{\mathbf{x}}^{(k)})} \times \right. \\
&\left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)+}(\mathbf{x}^{(1)})}} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)+}(\mathbf{x}^{(1)})}} \right) d\bar{\mathbf{x}}^{(k-1)} \right] \quad (4.17)
\end{aligned}$$

$$\begin{aligned}
\text{(d) } f_{X_{j-\tau}^{(k)+} | U_j^+}(\mathbf{x}^{(k)} | \mathbf{u}_j) &= \\
f_{X^{(k)+}(\mathbf{x}^{(k)})} &+ 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu u_j} \int_{S^{(1)}} \dots \int_{S^{(k-1)}} f_{\bar{X}^{(k)+}(\bar{\mathbf{x}}^{(k)})} \times \right. \\
&\left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)+}(\mathbf{x}^{(1)})}} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)+}(\mathbf{x}^{(1)})}} \right) d\bar{\mathbf{x}}^{(k-1)} \right] \quad (4.18)
\end{aligned}$$

**Proof.**

To prove part (a), we treat the cases  $k = 1$  and  $k > 1$  separately, since the proof of the latter depends on that of the former.

For  $k = 1$ , Eq. (4.15) follows from Eq. (4.3) in view of Theorem 4.2 and the fact that  $e^{i2\pi\nu} = 1$ . For  $1 < k \leq N$ , we have by construction of  $\text{MARM}^+$  processes,

$$\begin{aligned}
f_{\bar{X}_{j+\tau}^{(k)+} | U_j^+}(\bar{\mathbf{x}}^{(k)} | \mathbf{u}_j) &= f_{X_{j+\tau}^{(1)+} | U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j) f_{X_{j+\tau}^{(2)+}, \dots, X_{j+\tau}^{(k)+} | X_{j+\tau}^{(1)+}, U_j^+}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)} | \mathbf{x}^{(1)}, \mathbf{u}_j) \\
&= f_{X_{j+\tau}^{(1)+} | U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j) f_{X_{j+\tau}^{(2)+}, \dots, X_{j+\tau}^{(k)+} | X_{j+\tau}^{(1)+}}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)} | \mathbf{x}^{(1)}) \quad (4.19) \\
&= f_{X_{j+\tau}^{(1)+} | U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j) \frac{f_{X_{j+\tau}^{(1)+}, X_{j+\tau}^{(2)+}, \dots, X_{j+\tau}^{(k)+}}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)})}{f_{X_{j+\tau}^{(1)+}}(\mathbf{x}^{(1)})}
\end{aligned}$$

Eq. (4.15) for  $1 < k \leq N$  follows by substituting the representation of  $f_{X_{j+\tau}^{(1)+} | U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j)$  for  $k = 1$  in Eq. (4.15) into Eq. (4.19).

The proof of Part (b) is similar to that of part (a), but with negative  $\tau$ . Eq. (4.16) for  $k = 1$  follows from Eq. (4.3) in view of Theorem 4.2 and the fact that  $e^{i2\pi\nu} = 1$ . For  $1 < k \leq N$ , we have by construction of  $\text{MARM}^+$  processes,

$$\begin{aligned}
f_{\bar{X}_{j-\tau}^{(k)+}|U_j^+}(\bar{\mathbf{x}}^{(k)} | \mathbf{u}_j) &= f_{X_{j-\tau}^{(1)+}|U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j) f_{X_{j-\tau}^{(2)+}, \dots, X_{j-\tau}^{(k)+}|X_{j-\tau}^{(1)+}, U_j^+}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)} | \mathbf{x}^{(1)}, \mathbf{u}_j) \\
&= f_{X_{j-\tau}^{(1)+}|U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j) f_{X_{j-\tau}^{(2)+}, \dots, X_{j-\tau}^{(k)+}|X_{j-\tau}^{(1)+}}(\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)} | \mathbf{x}^{(1)}) \quad (4.20) \\
&= f_{X_{j-\tau}^{(1)+}|U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j) \frac{f_{X_{j-\tau}^{(1)+}, X_{j-\tau}^{(2)+}, \dots, X_{j-\tau}^{(k)+}}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)})}{f_{X_{j-\tau}^{(1)+}}(\mathbf{x}^{(1)})}
\end{aligned}$$

Eq. (4.16) for  $1 < k \leq N$  follows by substituting the representation of  $f_{X_{j-\tau}^{(1)+}|U_j^+}(\mathbf{x}^{(1)} | \mathbf{u}_j)$  for  $k = 1$  in Eq. (4.16) into Eq. (4.20).

To prove part (c), write

$$f_{X_{j+\tau}^{(k)+}|U_j^+}(\mathbf{x}^{(k)} | \mathbf{u}_j) = \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}_{j+\tau}^{(k)+}|U_j^+}(\bar{\mathbf{x}}^{(k)} | \mathbf{u}_j) d\bar{\mathbf{x}}^{(k-1)}. \quad (4.21)$$

Eq. (4.17) readily follows by substituting Eq. (4.15) into Eq. (4.21).

Finally, to prove part (d), write

$$f_{X_{j-\tau}^{(k)+}|U_j^+}(\mathbf{x}^{(k)} | \mathbf{u}_j) = \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}_{j-\tau}^{(k)+}|U_j^+}(\bar{\mathbf{x}}^{(k)} | \mathbf{u}_j) d\bar{\mathbf{x}}^{(k-1)}. \quad (4.22)$$

Eq. (4.18) readily follows by substituting Eq. (4.16) into Eq. (4.22).  $\square$

### Corollary 4.3

For  $k = 1$  in Theorem 4.3, we have the following formulas as special cases:

$$f_{X_{j+\tau}^{(1)+}|U_j^+}(\mathbf{x} | \mathbf{u}) = f_{X^{(1)+}}(\mathbf{x}) + 2f_{X^{(1)+}}(\mathbf{x}) \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u} \times \left( \xi e^{i2\pi\nu \xi F_{X^{(1)+}}(\mathbf{x})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)+}}(\mathbf{x})} \right) \right] \quad (4.23)$$

$$f_{X_{j-\tau}^{(1)+}|U_j^+}(\mathbf{x} | \mathbf{u}) = f_{X^{(1)+}}(\mathbf{x}) + 2f_{X^{(1)+}}(\mathbf{x}) \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u} \times \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)+}}(\mathbf{x})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)+}}(\mathbf{x})} \right) \right] \quad (4.24)$$

$\square$

### Proposition 4.1

For  $\tau > 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $\mathbf{x}^{(k)} \in S^{(k)}$ ,  $k \geq 1$ ,

$$\begin{aligned}
F_{X_{j+\tau}^{(k)+}|U_j^+}(\mathbf{x}^{(k)} | \mathbf{u}_j) &= \\
&F_{X^{(k)+}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)+}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\
&\quad \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)+}}(\mathbf{y}^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)+}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right] \quad (4.25)
\end{aligned}$$

$$\begin{aligned}
F_{X_{j-\tau}^{(k)+} | U_j^+}(\mathbf{x}^{(k)} | \mathbf{u}_j) = \\
F_{X^{(k)+}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu \mathbf{u}_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\tilde{X}^{(k)+}}(\vec{\mathbf{y}}^{(k)}) \times \right. \\
\left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)+}}(\mathbf{y}^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)+}}(\mathbf{y}^{(1)})} \right) d\vec{\mathbf{y}}^{(k)} \right] \quad (4.26)
\end{aligned}$$

**Proof.**

Eqs. (4.25) and (4.26) follow by integrating Eqs. (4.17) and (4.18), respectively, in Theorem 4.3.  $\square$

#### Corollary 4.4

For  $k = 1$  in Proposition 4.1, we have the following formulas as special cases:

$$\begin{aligned}
F_{X_{j+\tau}^{(1)+} | U_j^+}(\mathbf{x} | \mathbf{u}) = F_{X^{(1)+}}(\mathbf{x}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1}, j+\tau} (i2\pi\nu) e^{-i2\pi\nu \mathbf{u}} \int_{-\infty}^{\mathbf{x}} f_{X^{(1)+}}(\mathbf{y}) \times \right. \\
\left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)+}}(\mathbf{y})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)+}}(\mathbf{y})} \right) d\mathbf{y} \right] \quad (4.27)
\end{aligned}$$

$$\begin{aligned}
F_{X_{j-\tau}^{(1)+} | U_j^+}(\mathbf{x} | \mathbf{u}) = F_{X^{(1)+}}(\mathbf{x}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu \mathbf{u}} \int_{-\infty}^{\mathbf{x}} f_{X^{(1)+}}(\mathbf{y}) \times \right. \\
\left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)+}}(\mathbf{y})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)+}}(\mathbf{y})} \right) d\mathbf{y} \right] \quad (4.28)
\end{aligned}$$

$\square$

#### Proposition 4.2

For  $\tau > 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $k \geq 1$ ,

$$\mathbf{E}[X_{j+\tau}^{(k)+} | U_j^+ = u_j] = \mu_{X^{(k)+}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1}, j+\tau} (i2\pi\nu) e^{-i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu) \right] \quad (4.29)$$

$$\mathbf{E}[X_{j-\tau}^{(k)+} | U_j^+ = u_j] = \mu_{X^{(k)+}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(i2\pi\nu) \right] \quad (4.30)$$

**Proof.**

For Eq. (4.29),

$$\begin{aligned}
\mathbf{E}[X_{j+\tau}^{(k)+} | U_j^+ = u_j] &= \mathbf{E} \left[ \mathbf{E} \left[ X_{j+\tau}^{(k)+} | U_{j+\tau}^+ = v, \vec{W}_{j+\tau}^{(k-1)} = \vec{w}_{j+\tau}^{(k-1)} \right] \right] \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j+\tau}^{(k-1)}) f_{U_{j+\tau}^+ | U_j^+}(v | u_j) f_{\vec{W}_{j+\tau}^{(k-1)}}(\vec{w}_{j+\tau}^{(k-1)}) dv d\vec{w}_{j+\tau}^{(k-1)} \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j+\tau}^{(k-1)}) f_{U_{j+\tau}^+ | U_j^+}(v | u_j) dv d\vec{w}_{j+\tau}^{(k-1)} \\
&= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1}, j+\tau} (i2\pi\nu) e^{-i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)
\end{aligned}$$

where the multiple integrals on the right-hand side above consists of  $n$  integrals. Here, the third equation uses the fact that  $f_{\vec{W}_{j+\tau}^{(k-1)}}(\vec{w}_{j+\tau}^{(k-1)}) = 1$  in the multiple integral above, and the last

equation follows from Eq. (4.13). Eq. (4.29) now follows with the aid of Lemma 4.1 and Lemma 4.2.

In a similar vein, for Eq. (4.30)

$$\begin{aligned}
\mathbf{E}[X_{j-\tau}^{(k)+} | U_j^+ = u_j] &= \mathbf{E}\left[\mathbf{E}\left[X_{j-\tau}^{(k)+} | U_{j-\tau}^+ = v, \vec{W}_{j-\tau} = \vec{w}_{j-\tau}^{(k-1)}\right]\right] \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j-\tau}^{(k-1)}) f_{U_{j-\tau}^+ | U_j^+}(v | u_j) f_{\vec{W}_{j-\tau}}(\vec{w}_{j-\tau}^{(k-1)}) dv d\vec{w}_{j-\tau}^{(k-1)} \\
&= \int_0^1 \cdots \int_0^1 D^{(k)}(v, \vec{w}_{j-\tau}^{(k-1)}) f_{U_{j-\tau}^+ | U_j^+}(v | u_j) dv d\vec{w}_{j-\tau}^{(k-1)} \\
&= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu) e^{i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(i2\pi\nu)
\end{aligned}$$

Eq. (4.30) now follows with the aid of Lemma 4.1 and Lemma 4.2.  $\square$

## 4.2. Transition Structure of MARM<sup>-</sup> Processes

The following theorem is a fundamental representation for the transition density function,  $f_{U_{j+\tau}^- | U_j^-}(v | u)$ , of MARM<sup>-</sup> background processes.

### Theorem 4.4

$\{U_n^-\}$  is Markovian with non-stationary transition density  $f_{U_{j+\tau}^- | U_j^-}(v | u)$ , where for  $\tau > 0$  and  $0 \leq u, v < 1$ ,

$$f_{U_{j+\tau}^- | U_j^-}(v | u) = \begin{cases} 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu) e^{i2\pi\nu(v-u)} \right], & j \text{ even, } \tau \text{ even} \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu) e^{i2\pi\nu(-v-u)} \right], & j \text{ even, } \tau \text{ odd} \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu) e^{i2\pi\nu(-v+u)} \right], & j \text{ odd, } \tau \text{ even} \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1,j+\tau}}(i2\pi\nu) e^{i2\pi\nu(v+u)} \right], & j \text{ odd, } \tau \text{ odd} \end{cases} \quad (4.31)$$

$$f_{U_{j-\tau}^- | U_j^-}(v | u) = \begin{cases} 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu) e^{i2\pi\nu(u-v)} \right], & j \text{ even, } \tau \text{ even} \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu) e^{i2\pi\nu(u+v)} \right], & j \text{ even, } \tau \text{ odd} \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu) e^{i2\pi\nu(-u+v)} \right], & j \text{ odd, } \tau \text{ even} \\ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1,j}}(i2\pi\nu) e^{i2\pi\nu(-u-v)} \right], & j \text{ odd, } \tau \text{ odd} \end{cases} \quad (4.32)$$

**Proof.**

To prove Eq. (4.31), we use Theorem 2 in [25] to write

$$f_{U_{j+\tau}^- | U_j^-}(v | u) = \begin{cases} \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu(v-u)}, & j \text{ even, } \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu(-v-u)}, & j \text{ even, } \tau \text{ odd} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu(-v+u)}, & j \text{ odd, } \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu(v+u)}, & j \text{ odd, } \tau \text{ odd} \end{cases} \quad (4.33)$$

Eq. (4.31) now follows from Eq. (4.33) with the aid of Lemma 4.1 and by noticing the fact that  $\tilde{f}_{S_{j+1, j+\tau}}(0) = \tilde{f}_{S_{j-\tau+1, j}}(0) = 1$ .

Eq. (4.32) is proved in a similar way. It follows from Eq.(4.1), Theorem 4.2 and Eq.(4.31) that

$$f_{U_{j-\tau}^- | U_j^-}(v | u) = \begin{cases} \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu(u-v)}, & j \text{ even, } \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu(u+v)}, & j \text{ even, } \tau \text{ odd} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu(-u+v)}, & j \text{ odd, } \tau \text{ even} \\ \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu(-u-v)}, & j \text{ odd, } \tau \text{ odd} \end{cases} \quad (4.34)$$

Eq. (4.32) now follows from Eq. (4.34) with the aid of Lemma 4.1 and the fact that  $\tilde{f}_{S_{j+1, j+\tau}}(0) = \tilde{f}_{S_{j+\tau+1, j}}(0) = 1$ .  $\square$

#### Theorem 4.5

Let  $\tau > 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $\bar{x}^{(k)} \in S^{(1)} \times \dots \times S^{(k)}$ ,  $x^{(k)} \in S^{(k)}$ ,  $k \geq 1$ .

$$\begin{aligned}
\text{(a) } f_{\bar{X}_{j+\tau}^{(k)-} | U_j^-}(\bar{x}^{(k)} | u_j) = & \\
& \left\{ \begin{aligned}
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ even, } \tau \text{ even} \\
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ even, } \tau \text{ odd} \\
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ odd, } \tau \text{ even} \\
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ odd, } \tau \text{ odd}
\end{aligned} \right. \tag{4.35}
\end{aligned}$$

$$\begin{aligned}
\text{(b) } f_{\bar{X}_{j-\tau}^{(k)-} | U_j^-}(\bar{x}^{(k)} | u_j) = & \\
& \left\{ \begin{aligned}
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ even, } \tau \text{ even} \\
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ even, } \tau \text{ odd} \\
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ odd, } \tau \text{ even} \\
& f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) \right] \right\}, \quad j \text{ odd, } \tau \text{ odd}
\end{aligned} \right. \tag{4.36}
\end{aligned}$$

$$\begin{aligned}
\text{(c) } f_{X_{j+\tau}^{(k)-} | U_j^-}(x^{(k)} | u_j) = & \\
& \left\{ \begin{aligned}
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ even, } \tau \text{ even} \\
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ even, } \tau \text{ odd} \\
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ odd, } \tau \text{ even} \\
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ odd, } \tau \text{ odd}
\end{aligned} \right.
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\text{(d) } f_{X_{j-\tau}^{(k)-} | U_j^-}(x^{(k)} | u_j) = & \\
& \left\{ \begin{aligned}
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ even, } \tau \text{ even} \\
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ even, } \tau \text{ odd} \\
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ odd, } \tau \text{ even} \\
& f_{X^{(k)-}}(x^{(k)}) + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{x}^{(k)}) \times \right. \\
& \quad \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(x^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x^{(1)})} \right) d\bar{x}^{(k-1)} \right], j \text{ odd, } \tau \text{ odd}
\end{aligned} \right.
\end{aligned} \tag{4.38}$$

**Proof.**

Similar to the proof of Theorem 4.3 with  $\text{MARM}^-$  random variables replacing their  $\text{MARM}^+$  counterparts.  $\square$

**Corollary 4.5**

For  $k = 1$  in Theorem 4.5, we have the following formulas as special cases:

$$f_{X_{j+\tau}^{(1)-} | U_j^-}(x | u_j) = \left\{ \begin{array}{l} f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\ \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ even, } \tau \text{ even} \\ f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\ \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ even, } \tau \text{ odd} \\ f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\ \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ odd, } \tau \text{ even} \\ f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \text{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\ \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ odd, } \tau \text{ odd} \end{array} \right. \quad (4.39)$$



$$\begin{aligned}
& f_{X_{j-\tau}^{(1)-} | U_j^-}(x | u_j) = \\
& \left\{ \begin{aligned}
& f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ even}, \tau \text{ even} \\
& f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ even}, \tau \text{ odd} \\
& f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ odd}, \tau \text{ even} \\
& f_{X^{(1)-}}(x) \left\{ 1 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1}, j} (i2\pi\nu) e^{-i2\pi\nu u_j} \times \right. \right. \\
& \quad \left. \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(x)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(x)} \right) \right] \right\}, j \text{ odd}, \tau \text{ odd}
\end{aligned} \right.
\end{aligned}$$

(4.40)

□

**Proposition 4.3**

For  $\tau > 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $\mathbf{x}^{(k)} \in S^{(k)}$ ,  $k \geq 1$ ,

$$F_{X_{j+\tau}^{(k)-} | U_j^-}(\mathbf{x}^{(k)} | \mathbf{u}_j) = \left\{ \begin{array}{l} F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ even}, \tau \text{ even} \\ \\ F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ even}, \tau \text{ odd} \\ \\ F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ odd}, \tau \text{ even} \\ \\ F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ odd}, \tau \text{ odd} \end{array} \right. \quad (4.41)$$

$$F_{X_{j-\tau}^{(k)-} | U_j^-}(\mathbf{x}^{(k)} | \mathbf{u}_j) = \left\{ \begin{array}{l} F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ even}, \tau \text{ even} \\ \\ F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ even}, \tau \text{ odd} \\ \\ F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ odd}, \tau \text{ even} \\ \\ F_{X^{(k)-}}(\mathbf{x}^{(k)}) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^{\mathbf{x}^{(k)}} \int_{S^{(1)}} \cdots \int_{S^{(k-1)}} f_{\bar{X}^{(k)-}}(\bar{\mathbf{y}}^{(k)}) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}}(\mathbf{y}^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(\mathbf{y}^{(1)})} \right) d\bar{\mathbf{y}}^{(k)} \right], j \text{ odd}, \tau \text{ odd} \end{array} \right. \quad (4.42)$$

**Proof.**

Eqs. (4.41) and (4.42) follow by integrating Eqs. (4.37) and (4.38), respectively, in Theorem 4.5.  $\square$

**Corollary 4.6**

For  $k = 1$  in Proposition 4.3, we have the following formulas as special cases:

$$F_{X_{j+\tau}^{(1)-}|U_j^-}(x | u_j) = \left\{ \begin{array}{l} F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ even}, \tau \text{ even} \\ \\ F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ even}, \tau \text{ odd} \\ \\ F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ odd}, \tau \text{ even} \\ \\ F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ odd}, \tau \text{ odd} \end{array} \right. \quad (4.43)$$

and

$$F_{X_{j-\tau}^{(1)-}|U_j^-}(x | u_j) = \left\{ \begin{array}{l} F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ even}, \tau \text{ even} \\ \\ F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ even}, \tau \text{ odd} \\ \\ F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ odd}, \tau \text{ even} \\ \\ F_{X^{(1)-}}(x) + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \int_{-\infty}^x f_{X^{(1)-}}(y) \times \right. \\ \left. \left( \xi e^{-i2\pi\nu\xi F_{X^{(1)-}}(y)} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}}(y)} \right) dy \right], j \text{ odd}, \tau \text{ odd} \end{array} \right. \quad (4.44)$$

□

**Proposition 4.4**

For  $\tau > 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $k \geq 1$ ,

$$\mathbf{E}[X_{j+\tau}^{(k)-} | U_j^- = u_j] = \begin{cases} \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu) \right], & j \text{ even, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{-i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(i2\pi\nu) \right], & j \text{ even, } \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(i2\pi\nu) \right], & j \text{ odd, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) e^{i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu) \right], & j \text{ odd, } \tau \text{ odd} \end{cases} \quad (4.45)$$

$$\mathbf{E}[X_{j-\tau}^{(k)-} | U_j^- = u_j] = \begin{cases} \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(i2\pi\nu) \right], & j \text{ even, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu) \right], & j \text{ even, } \tau \text{ odd} \\ \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(-i2\pi\nu) \right], & j \text{ odd, } \tau \text{ even} \\ \mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) e^{-i2\pi\nu u_j} \tilde{\mathcal{D}}^{(k)}(i2\pi\nu) \right], & j \text{ odd, } \tau \text{ odd} \end{cases} \quad (4.46)$$

**Proof.**

Similar to the proof of Proposition 4.2 with  $\text{MARM}^-$  random variables replacing their  $\text{MARM}^+$  counterparts. □

**5. The Autocorrelation and Cross-Correlation Structures of MARM Processes**

To derive the autocorrelation and cross-correlation structure of MARM processes, we need to derive the expectations of products of lagged foreground random variables of the form

$$\mathbf{E} \left[ X_j^{(m)} X_{j+\tau}^{(n)} \right] = \int_{S^{(m)}} \int_{S^{(n)}} x_j^{(m)} x_{j+\tau}^{(n)} f_{X_j^{(m)}, X_{j+\tau}^{(n)}}(x_j^{(m)}, x_{j+\tau}^{(n)}) dx_j^{(m)} dx_{j+\tau}^{(n)}, \quad (5.1)$$

for  $1 \leq m \leq n \leq N$ ,  $\tau > 0$ ,  $j \geq 0$ , as an ingredient of the correlation functions,  $\rho_{m,n}(j, \tau)$  of MARM processes  $\{X_j^{(m)}\}_{j=0}^{\infty}$  and  $\{X_j^{(n)}\}_{j=0}^{\infty}$ . Recall that cases of the form  $m = n$  correspond to autocorrelation functions, while cases of the form  $m \neq n$  correspond to cross-correlation functions. To this end, we derive conditional versions of these expectations and then uncondition

them, recalling the random-vector generating algorithm (Algorithm 2.1) in Section 2 and the MARM foreground-process generating algorithm (Algorithm 3.1) in Section 3.3.

The following lemma provides a simplified representation of the joint density of pairs of MARM random variables.

**Lemma 5.1**

For  $\tau \geq 0$ ,  $0 \leq u_j < 1$ ,  $j \geq 0$ , and  $\mathbf{x}^{(m)} \in \mathcal{S}^{(m)}$ ,  $\mathbf{x}^{(n)} \in \mathcal{S}^{(n)}$ ,  $1 \leq m < n \leq N$ ,

$$f_{\mathbf{X}_j^{(m)}, \mathbf{X}_{j+\tau}^{(n)} | U_j}(\mathbf{x}_j^{(m)}, \mathbf{x}_{j+\tau}^{(n)} | u_j) = f_{\mathbf{X}_j^{(m)} | U_j}(\mathbf{x}_j^{(m)} | u_j) f_{\mathbf{X}_{j+\tau}^{(n)} | U_j}(\mathbf{x}_{j+\tau}^{(n)} | u_j) \quad (5.2)$$

**Proof.**

By definition,

$$f_{\mathbf{X}_k^{(m)}, \mathbf{X}_{j+\tau}^{(n)} | U_j}(\mathbf{x}_j^{(m)}, \mathbf{x}_{j+\tau}^{(n)} | u_j) = f_{\mathbf{X}_j^{(m)} | U_j}(\mathbf{x}_j^{(m)} | u_j) f_{\mathbf{X}_{j+\tau}^{(n)} | U_j, \mathbf{X}_j^{(m)}}(\mathbf{x}_{j+\tau}^{(n)} | u_j, \mathbf{x}_j^{(m)}).$$

The lemma now follows since  $f_{\mathbf{X}_{j+\tau}^{(n)} | U_j, \mathbf{X}_j^{(m)}}(\mathbf{x}_{j+\tau}^{(n)} | u_j, \mathbf{x}_j^{(m)}) = f_{\mathbf{X}_{j+\tau}^{(n)} | U_j}(\mathbf{x}_{j+\tau}^{(n)} | u_j)$  by construction of MARM processes.  $\square$

### 5.1. Correlation Structure of MARM<sup>+</sup> Processes

In this section we derive general formulas for the autocorrelation and cross-correlation functions of MARM<sup>+</sup> processes. The following theorem generalizes distortion functions [Melamed (1999)] from ARM<sup>+</sup> processes to MARM<sup>+</sup> processes.

The next theorem exhibits computational formulas for correlation functions of MARM<sup>+</sup> processes.

**Theorem 5.1**

For  $1 \leq m \leq n \leq N$ ,  $\tau > 0$  and  $j \geq 0$ , the correlation functions  $\rho_{m,n}^+(j, \tau)$  are given by

$$\rho_{m,n}^+(j, \tau) = \frac{2}{\sigma_{\mathbf{X}^{(m)+}} \sigma_{\mathbf{X}^{(n)+}}} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{\mathcal{S}_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right] \quad (5.3)$$

In particular, for  $m = n$ , Eq. (5.3) reduces to

$$\rho_{n,n}^+(j, \tau) = \frac{2}{\sigma_{\mathbf{X}^{(n)+}}^2} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{\mathcal{S}_{j+1, j+\tau}}(i2\pi\nu) \right] \left| \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right|^2 \quad (5.4)$$

**Proof.**

We first show that for  $1 \leq m \leq n \leq N$ ,

$$\mathbf{E}[\mathbf{X}_j^{(m)+} \mathbf{X}_{j+\tau}^{(n)+}] = \mu_{\mathbf{X}^{(m)+}} \mu_{\mathbf{X}^{(n)+}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{\mathcal{S}_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right] \quad (5.5)$$

and in particular, for  $m = n$ , Eq. (5.5) reduces to

$$\mathbf{E}[\mathbf{X}_j^{(n)+} \mathbf{X}_{j+\tau}^{(n)+}] = \mu_{\mathbf{X}^{(n)+}}^2 + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{\mathcal{S}_{j+1, j+\tau}}(i2\pi\nu) \right] \left| \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right|^2. \quad (5.6)$$

We prove Eq. (5.5) in three cases.

For  $m = n = 1$ , Eq. (5.5) follows from Theorem 3 in [Melamed (1999)] by noting that  $\mathcal{D}^{(1)}$  is equivalent to the distortion  $D$  therein (see also Theorem 3 in Jagerman and Melamed (1992a)).

For  $m = 1$  and  $2 \leq n \leq N$ , we use Eqs. (4.7) and (4.8) to write

$$\mathbf{E} \left[ \mathbf{X}_j^{(1)+} \mathbf{X}_{j+\tau}^{(n)+} \mid U_j^+ = \mathbf{u}, U_{j+\tau}^+ = \mathbf{v}, \vec{W}_{j+\tau}^{(n-1)} = \vec{w}_{j+\tau}^{(n-1)} \right] = D^{(1)}(\mathbf{u}) D^{(n)}(\mathbf{v}, \vec{w}_{j+\tau}^{(n-1)}).$$

Consequently,

$$\begin{aligned} \mathbf{E} \left[ \mathbf{X}_j^{(1)+} \mathbf{X}_{j+\tau}^{(n)+} \right] &= \mathbf{E} \left[ \mathbf{E} \left[ \mathbf{X}_j^{(1)+} \mathbf{X}_{j+\tau}^{(n)+} \mid U_j^+, U_{j+\tau}^+, \vec{W}_{j+\tau}^{(n-1)} \right] \right] \\ &= \int_0^1 \cdots \int_0^1 D^{(1)}(\mathbf{u}) D^{(n)}(\mathbf{v}, \vec{w}_{j+\tau}^{(n-1)}) f_{U_j^+}(\mathbf{u}) f_{U_{j+\tau}^+ | U_j^+}(\mathbf{v} \mid \mathbf{u}) f_{\vec{W}_{j+\tau}^{(n-1)}}(\vec{w}_{j+\tau}^{(n-1)}) d\mathbf{u} d\mathbf{v} d\vec{w}_{j+\tau}^{(n-1)} \\ &= \int_0^1 \cdots \int_0^1 D^{(1)}(\mathbf{u}) D^{(n)}(\mathbf{v}, \vec{w}_{j+\tau}^{(n-1)}) f_{U_{j+\tau}^+ | U_j^+}(\mathbf{v} \mid \mathbf{u}) d\mathbf{u} d\mathbf{v} d\vec{w}_{j+\tau}^{(n-1)} \\ &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(1)}(i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \end{aligned}$$

where the multiple integrals on the right-hand side above consists of  $n + 1$  integrals. Here, the third equation uses the fact that  $f_{U_j^+}(\mathbf{u}) = 1$  and  $f_{\vec{W}_{j+\tau}^{(n-1)}}(\vec{w}_{j+\tau}^{(n-1)}) = 1$  in the multiple integral above, and the last equation follows from Eq. (4.13), Eq. (4.9). Eq. (5.5) now follows with the aid of Lemma 4.1 and Lemma 4.2 by noting that  $\tilde{f}_{S_{j+1, j+\tau}}(0) \tilde{\mathcal{D}}^{(1)}(0) \tilde{\mathcal{D}}^{(n)}(0) = \mu_{X^{(1)+}} \mu_{X^{(n)+}}$ .

For  $2 \leq m \leq n \leq N$  note that

$$\begin{aligned} \mathbf{E} \left[ \mathbf{X}_j^{(m)+} \mathbf{X}_{j+\tau}^{(n)+} \mid U_j^+ = \mathbf{u}, U_{j+\tau}^+ = \mathbf{v}, \vec{W}_j^{(m-1)} = \vec{w}_j^{(m-1)}, \vec{W}_{j+\tau}^{(n-1)} = \vec{w}_{j+\tau}^{(n-1)} \right] \\ = D^{(m)}(\mathbf{u}, \vec{w}_j^{(m-1)}) D^{(n)}(\mathbf{v}, \vec{w}_{j+\tau}^{(n-1)}) \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbf{E} \left[ \mathbf{X}_j^{(m)+} \mathbf{X}_{j+\tau}^{(n)+} \right] &= \mathbf{E} \left[ \mathbf{E} \left[ \mathbf{X}_j^{(m)+} \mathbf{X}_{j+\tau}^{(n)+} \mid U_j^+ = \mathbf{u}, U_{j+\tau}^+ = \mathbf{v}, \vec{W}_j^{(m-1)} = \vec{w}_j^{(m-1)}, \vec{W}_{j+\tau}^{(n-1)} = \vec{w}_{j+\tau}^{(n-1)} \right] \right] \\ &= \int_0^1 \cdots \int_0^1 D^{(m)}(\mathbf{u}, \vec{w}_j^{(m-1)}) D^{(n)}(\mathbf{v}, \vec{w}_{j+\tau}^{(n-1)}) f_{U_j^+}(\mathbf{u}) f_{U_{j+\tau}^+ | U_j^+}(\mathbf{v} \mid \mathbf{u}) \times \\ &\quad f_{\vec{W}_j^{(m-1)}}(\vec{w}_j^{(m-1)}) f_{\vec{W}_{j+\tau}^{(n-1)}}(\vec{w}_{j+\tau}^{(n-1)}) d\mathbf{u} d\mathbf{v} d\vec{w}_j^{(m-1)} d\vec{w}_{j+\tau}^{(n-1)} \end{aligned}$$

where the right-hand side above consists of  $m + n$  integrals. Evaluating the multiple integrals above yields

$$\mathbf{E} \left[ \mathbf{X}_j^{(m)+} \mathbf{X}_{j+\tau}^{(n)+} \right] = \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu)$$

where we again make use of Eqs. (4.13) and (4.9), and the facts that  $f_{U_j^+}(\mathbf{u}) = 1$ ,  $f_{\vec{W}_j^{(m-1)}}(\vec{w}_j^{(m-1)}) = 1$  and  $f_{\vec{W}_{j+\tau}^{(n-1)}}(\vec{w}_{j+\tau}^{(n-1)}) = 1$  in the multiple integral above. Eq. (5.5) now

follows by applying Lemma 4.1 to the representation above and noting that  $\tilde{f}_{S_{j+1, j+\tau}}(0) \tilde{\mathcal{D}}^{(m)}(0) \tilde{\mathcal{D}}^{(n)}(0) = \mu_{X^{(m)+}} \mu_{X^{(n)+}}$  with the aid of Lemma 4.2.

Next, Eq. (5.6) follows from Eq. (5.5) by observing that  $\tilde{\mathcal{D}}^{(k)}(i2\pi\nu)$  and  $\tilde{\mathcal{D}}^{(k)}(-i2\pi\nu)$  are complex conjugates by Lemma 4.1. Finally, Eqs. (5.3) and (5.4) follow readily by substituting Eqs. (5.5) and (5.6) into Eqs. (2.2) and (2.1), respectively.  $\square$

Note that when the innovation sequence,  $\{V_n\}$ , is iid, it follows that the correlation functions  $\rho_{m,n}^+(j, \tau) = \rho_{m,n}^+(\tau)$  are stationary and depend only on the lag  $\tau$ .

## 5.2. Correlation Structure of MARM<sup>-</sup> Processes

The next theorem exhibits computational formulas for correlation functions of MARM<sup>-</sup> processes.

### Theorem 5.2

For  $1 \leq m \leq n \leq N$ ,  $\tau > 0$  and  $j \geq 0$ , the correlation functions  $\rho_{m,n}^-(j, \tau)$  are given by

$$\rho_{m,n}^-(j, \tau) = \begin{cases} \frac{2}{\sigma_{X^{(m)-}} \sigma_{X^{(n)-}}} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right], & j \text{ even}, \tau \text{ even} \\ \frac{2}{\sigma_{X^{(m)-}} \sigma_{X^{(n)-}}} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right], & j \text{ even}, \tau \text{ odd} \\ \frac{2}{\sigma_{X^{(m)-}} \sigma_{X^{(n)-}}} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(-i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right], & j \text{ odd}, \tau \text{ even} \\ \frac{2}{\sigma_{X^{(m)-}} \sigma_{X^{(n)-}}} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \tilde{\mathcal{D}}^{(m)}(-i2\pi\nu) \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right], & j \text{ odd}, \tau \text{ odd} \end{cases} \quad (5.7)$$

In particular, for  $m = n$ , Eq. (5.7) reduces to

$$\rho_{n,n}^-(j, \tau) = \begin{cases} \frac{2}{\sigma_{X^{(n)-}}^2} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \right] \left| \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right|^2, & j \text{ even}, \tau \text{ even} \\ \frac{2}{\sigma_{X^{(n)-}}^2} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \left( \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right)^2 \right], & j \text{ even}, \tau \text{ odd} \\ \frac{2}{\sigma_{X^{(n)-}}^2} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \right] \left| \tilde{\mathcal{D}}^{(n)}(i2\pi\nu) \right|^2, & j \text{ odd}, \tau \text{ even} \\ \frac{2}{\sigma_{X^{(n)-}}^2} \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \left( \tilde{\mathcal{D}}^{(n)}(-i2\pi\nu) \right)^2 \right], & j \text{ odd}, \tau \text{ odd} \end{cases} \quad (5.8)$$

**Proof.**

Similar to the proof of Theorem 5.1 with  $\text{MARM}^-$  random variables replacing their  $\text{MARM}^+$  counterparts.  $\square$

## 6. The MARM Fitting Methodology

In this section we formulate the general MARM fitting methodology. A practical empirical version of it will be described in some detail in the aforementioned companion paper (Part II), Melamed and Zhao (2011). The general MARM fitting problem can be formulated as the following optimization problem.

### Problem 6.1 (General MARM Fitting Problem)

Given a multivariate distribution  $\hat{F}_{\bar{X}}$  and a set of stationary correlation functions  $\hat{\rho}_{m,n}(\tau)$ , find an iid innovation probability law,  $f_{\{V_n^*\}}$ , and a stitching parameter  $\xi^*$  such that

$$(f_{\{V_n^*\}}, \xi^*) = \arg \min_{(f_{\{V_n\}}, \xi)} \{g(f_{\{V_n\}}, \xi)\}$$

where  $g(f_{\{V_n\}}, \xi)$  is an objective function of the form

$$g(f_{\{V_n\}}, \xi) = \sum_{m=1}^N \sum_{n=m}^N \sum_{\tau=1}^{S(m,n)} a_{m,n}(\tau) [\rho_{m,n}(\tau) - \hat{\rho}_{m,n}(\tau)]^2, \quad (6.1)$$

$S(m, n)$  is the maximal correlation lag for pair  $(m, n)$ , and  $0 \leq a_{m,n}(\tau) \leq 1$  are weights.

The general MARM fitting problem aims to satisfy the first two goodness-of-fit criteria in Section 1 (recall that the third one is a subjective judgment).

- 1) It automatically satisfies the first criterion, because every MARM process constructed from a multivariate distribution has the latter's marginal distributions by construction. This assertion follows from the following facts. First, by Lemma 3.1 (General Iterated Uniformity), all MARM background processes are Markovian, and their marginal distribution is uniform on  $[0,1)$  regardless of the probability law of the innovations  $\{V_n\}$  selected; furthermore, stitching transformations preserve uniformity by Lemma 3.2. And second, the Inversion Method permits us, in principle, to transform any uniform random variable  $U$  to others with arbitrary distribution  $F$  through  $F^{-1}(U)$ , and Algorithm 2.1 (RVGA) extends this approach to arbitrary multivariate distributions. Thus, any MARM foreground process obtained by applying the Inversion Method to a stitched background process is guaranteed to have the (given) multivariate distribution, regardless of the innovation sequence,  $\{V_n\}$ , and stitching parameter,  $\xi$ , selected.
- 2) Minimizing the objective function in Eq. (6.1) satisfies the second criterion.

Our solution approach to Problem 6.1 is the *CSLO* (*Comprehensive Search Local Optimization*) algorithm, which generalizes the *GSLO* algorithm in [Jelenkovic and Melamed (1995a, 1995b)] from ARM processes to MARM processes. As the name suggests, CSLO consists of two sequential stages:

1. **Comprehensive Search Stage.** This stage performs a comprehensive search at some prescribed granularity over pairs  $(f_{\{V_n\}}, \xi)$ , which results in a set of  $B$  "best candidate



models” (those with the smallest values of the associated objective function). This set provides  $|B|$  “promising” initial models as input to the local optimization of the next stage.

2. **Local Optimization Stage.** This stage performs optimization on each initial model by minimizing its objective function, using the steepest descent method [Bazaraa et al. (1993)]. This optimization requires the computation of the derivatives of the objective function with respect to the aforementioned search parameters.

We point out that the objective function in Eq. (6.1) does not contain penalty terms due to the number of parameters or observations involved; see [Vapnik (1998), Massart (2007)]. The reason is that in this paper, each fitting run is associated with a fixed number of parameters and statistical signature. More specifically, the MARM fitting methodology utilizes a prescribed but fixed number of parameters to parameterize the entire search space of pairs,  $(f_{\{V_n\}}, \xi)$ ; furthermore, when the statistical signature is empirically based, that signature is obtained from a given, but fixed, set of empirical observations (if the empirical sample changes, the fitting procedure is simply rerun). For more details, see [Jelenkovic and Melamed (1995a, 1995b)].

## 7. The MARM Forecasting Methodology

In this section, we describe the MARM forecasting methodology which consists of point estimators and confidence intervals of foreground MARM processes. More specifically, for each  $1 \leq k \leq N$ , time index  $j \geq 0$  and lag  $\tau > 0$ , and given the observed history  $\{\vec{X}_i^{(k)} : i \leq j\}$ ,

- The point estimator of  $X_{j+\tau}^{(k)}$  given  $\vec{X}_j^{(k)}$  is an estimator of  $E[X_{j+\tau}^{(k)} | \vec{X}_j^{(k)}]$ .
- The  $1 - \alpha$  confidence interval for  $X_{j+\tau}^{(k)}$  is computed from an estimator of  $F_{X_{j+\tau}^{(k)} | \vec{X}_j^{(k)}}$

To this end, the MARM forecasting methodology makes use of estimators of conditional densities of the form  $f_{X_{j+\tau}^{(k)} | \vec{X}_j^{(k)}}$  and  $f_{X_{j-\tau}^{(k)} | U_j}$ ,  $1 \leq k \leq N$ . Note that the densities  $f_{X_{j+\tau}^{(k)} | \vec{X}_j^{(k)}}$  and  $f_{X_{j-\tau}^{(k)} | U_j}$  pertain to time-reversed MARM-related processes [Kelly (1979)].

### 7.1. Point Estimators and Confidence Intervals for MARM Foreground Processes

In this section, we describe the development of point estimators and confidence intervals for foreground processes using estimated densities, denoted by  $\hat{f}_{X_{j+\tau}^{(k)} | \vec{X}_j^{(k)}}$ ,  $1 \leq k \leq N$ . Note that for  $\tau \neq 0$  the construction of MARM processes ensures that

$$\hat{f}_{X_{j+\tau}^{(k)} | \vec{X}_j^{(k)}} = \hat{f}_{X_{j+\tau}^{(k)} | X_j^{(1)}}, \quad 1 \leq k \leq N. \quad (7.1)$$

To this end, we need to supplement conditioning on  $\vec{X}_j^{(k)}$  by conditioning on  $U_j$  as an intermediary step. However, while  $X_j^{(1)}$  and  $U_j$  are related by Eq. (3.10), the former does not determine uniquely the latter. The problem stems from the fact that the stitching transformation (3.7) is not one-to-one; more specifically, for a given stitched value,  $F_{X^{(1)}}(x_j^{(1)})$ , there are generally two corresponding background values,  $u_j^{(1)}$  and  $u_j^{(2)}$ , which solve Eq. (3.7), namely,

$$\mathbf{u}_j^{(1)} = \xi \mathbf{F}_{\mathbf{X}^{(1)}}(\mathbf{x}_j^{(1)}), \quad \mathbf{u}_j^{(2)} = 1 - (1 - \xi) \mathbf{F}_{\mathbf{X}^{(1)}}(\mathbf{x}_j^{(1)}). \quad (7.2)$$

Thus, we need additional probabilistic information pertaining to the two solutions above. To this end, we postulate a simple discrete conditional distribution for the two solutions, say,

$$\Pr\{U_j = \mathbf{u}_j^{(1)} \mid \mathbf{X}_j^{(k)} = \mathbf{x}_j^{(k)}\} = p^{(k)} \quad \text{and} \quad \Pr\{U_j = \mathbf{u}_j^{(2)} \mid \mathbf{X}_j^{(k)} = \mathbf{x}_j^{(k)}\} = 1 - p^{(k)}, \quad (7.3)$$

where  $0 \leq p^{(k)} \leq 1$ ,  $1 \leq k \leq N$ , are user-selected parameters to be referred to as the *mixing parameter* in the  $k$ -th dimension, and their selection is described in Section 7.2. Accordingly, in view of Eq. (7.1), the MARM forecasting methodology uses a convex (probabilistic) combination of transition densities of the form

$$\hat{\mathbf{f}}_{\mathbf{X}_{j\pm\tau}^{(k)} \mid \mathbf{X}_j^{(1)}}(\mathbf{y} \mid \mathbf{x}) = p^{(k)} \mathbf{f}_{\mathbf{X}_{j\pm\tau}^{(k)} \mid U_j}(\mathbf{y} \mid \mathbf{u}_j^{(1)}) + (1 - p^{(k)}) \mathbf{f}_{\mathbf{X}_{j\pm\tau}^{(k)} \mid U_j}(\mathbf{y} \mid \mathbf{u}_j^{(2)}) \quad (7.4)$$

where the densities  $\mathbf{f}_{\mathbf{X}_{j\pm\tau}^{(k)} \mid U_j}(\mathbf{y} \mid \mathbf{u})$  are given in Eqs. (4.17), (4.18), (4.37) and (4.38).

Let  $\hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)} \mid \vec{\mathbf{X}}_j^{(k)}]$  be the conditional expectation induced by the conditional density  $\hat{\mathbf{f}}_{\mathbf{X}_{j\pm\tau}^{(k)} \mid \vec{\mathbf{X}}_j^{(k)}}(\mathbf{y} \mid \mathbf{x})$  from Eq. (7.4). The following proposition provides the structure of the requisite point estimators.

**Proposition 7.1**

For  $\tau > 0$ ,  $j \geq 0$  and  $\vec{\mathbf{x}}^{(k)} \in \mathcal{S}^{(1)} \times \dots \times \mathcal{S}^{(k)}$ ,  $1 \leq k \leq N$ ,

$$\begin{aligned} \hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)} \mid \vec{\mathbf{X}}_j^{(k)} = \vec{\mathbf{x}}_j^{(k)}] &= \hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)+} \mid \mathbf{X}_j^{(1)+} = \mathbf{x}_j^{(1)}] = \\ & p^{(k)} \mathbf{E}[\mathbf{X}_{j\pm\tau}^{(k)} \mid U_j = \mathbf{u}_j^{(1)}] + (1 - p^{(k)}) \mathbf{E}[\mathbf{X}_{j\pm\tau}^{(k)} \mid U_j = \mathbf{u}_j^{(2)}] \end{aligned} \quad (7.5)$$

**Proof.**

Immediate from Eqs. (7.1) and (7.4). □

The next two propositions present computational formulas for the point estimators  $\hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)+} \mid \vec{\mathbf{X}}_j^{(k)+}]$  and  $\hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)-} \mid \vec{\mathbf{X}}_j^{(k)-}]$ .

**Proposition 7.2**

For  $\tau > 0$ ,  $j \geq 0$  and  $\vec{\mathbf{x}}^{(k)} \in \mathcal{S}^{(1)} \times \dots \times \mathcal{S}^{(k)}$ ,  $1 \leq k \leq N$ ,

$$\begin{aligned} \hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)+} \mid \vec{\mathbf{X}}_j^{(k)+} = \vec{\mathbf{x}}_j^{(k)}] &= \hat{\mathbf{E}}[\mathbf{X}_{j\pm\tau}^{(k)+} \mid \mathbf{X}_j^{(1)+} = \mathbf{x}_j^{(1)}] = \\ & \mu_{\mathbf{X}^{(k)+}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{\mathbf{f}}_{\mathcal{S}_{j+1, j+\tau}} \left( i2\pi\nu \left( p^{(k)} e^{-i2\pi\nu \mathbf{u}_j^{(1)}} + (1 - p^{(k)}) e^{-i2\pi\nu \mathbf{u}_j^{(2)}} \right) \times \right. \right. \\ & \left. \left. \int_{\mathcal{S}^{(k)}} \dots \int_{\mathcal{S}^{(1)}} \mathbf{x}^{(k)} \mathbf{f}_{\vec{\mathbf{X}}^{(k)+}}(\vec{\mathbf{x}}^{(k)}) \left( \xi e^{i2\pi\nu \xi \mathbf{F}_{\mathbf{X}^{(1)+}}(\mathbf{x}^{(1)})} + (1 - \xi) e^{-i2\pi\nu(1-\xi) \mathbf{F}_{\mathbf{X}^{(1)+}}(\mathbf{x}^{(1)})} \right) \mathbf{d}\vec{\mathbf{x}}^{(k)} \right] \end{aligned} \quad (7.6)$$

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j-\tau}^{(k)+} | \vec{X}_j^{(k)+} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j-\tau}^{(k)+} | X_j^{(1)+} = x_j^{(1)}] = \\
&\mu_{X^{(k)+}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}} (i2\pi\nu) \left( p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)+}(\vec{x}^{(k)})} \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)+}(x^{(1)})}} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)+}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.7}$$

**Proof.**

It follows from Eqs. (7.5) and (4.17) that

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j+\tau}^{(k)+} | \vec{X}_j^{(k)+} = \vec{x}_j^{(k)}] &= \sum_{\nu=-\infty}^{\infty} \tilde{f}_{S_{j+1, j+\tau}} (i2\pi\nu) \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)+}(\vec{x}^{(k)})} \times \\
&\quad \left\{ p^{(k)} \left( \xi e^{i2\pi\nu[\xi F_{X^{(1)+}(x^{(1)})} - u_j^{(1)}]} + (1-\xi) e^{-i2\pi\nu[(1-\xi) F_{X^{(1)+}(x^{(1)})} + u_j^{(1)}]} \right) \right\} + \\
&\quad \left. (1-p^{(k)}) \left( \xi e^{i2\pi\nu[\xi F_{X^{(1)+}(x^{(1)})} - u_j^{(2)}]} + (1-\xi) e^{-i2\pi\nu[(1-\xi) F_{X^{(1)+}(x^{(1)})} + u_j^{(2)}]} \right) \right\} d\vec{x}^{(k)}
\end{aligned} \tag{7.8}$$

Eq. (7.6) is the reduced form of Eq.(7.8) with the aid of Lemma 4.1 and Lemma 4.2. Eq.(7.7) is proved in a similar way.  $\square$

### Proposition 7.3

Let  $\tau > 0$ ,  $j \geq 0$  and  $\vec{x}^{(k)} \in S^{(1)} \times \cdots \times S^{(k)}$ ,  $1 \leq k \leq N$ .

(a.1) For  $j$  even and  $\tau$  even,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}} (i2\pi\nu) \left( p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{-i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.9}$$

(a.2) For  $j$  even and  $\tau$  odd,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}} (i2\pi\nu) \left( p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{-i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \cdots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.10}$$

(a.3) For  $j$  odd and  $\tau$  even,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \left( p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \dots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.11}$$

(a.4) For  $j$  odd and  $\tau$  odd,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j+\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j+1, j+\tau}}(i2\pi\nu) \left( p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \dots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.12}$$

(b.1) For  $j$  even and  $\tau$  even,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) \left( p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \dots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{-i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.13}$$

(b.2) For  $j$  even and  $\tau$  odd,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) \left( p^{(k)} e^{i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \dots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.14}$$

(b.3) For  $j$  odd and  $\tau$  even,

$$\begin{aligned}
\hat{\mathbf{E}}[X_{j-\tau}^{(k)-} | \vec{X}_j^{(k)-} = \vec{x}_j^{(k)}] &= \hat{\mathbf{E}}[X_{j-\tau}^{(k)-} | X_j^{(1)-} = x_j^{(1)}] = \\
&\mu_{X^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \operatorname{Re} \left[ \tilde{f}_{S_{j-\tau+1, j}}(i2\pi\nu) \left( p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{-i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \dots \int_{S^{(1)}} x^{(k)} f_{\vec{X}^{(k)-}(\vec{x}^{(k)})} \left( \xi e^{i2\pi\nu \xi F_{X^{(1)-}(x^{(1)})}} + (1-\xi) e^{-i2\pi\nu(1-\xi) F_{X^{(1)-}(x^{(1)})}} \right) d\vec{x}^{(k)} \right]
\end{aligned} \tag{7.15}$$

(b.4) For  $j$  odd and  $\tau$  odd,

$$\begin{aligned}
\hat{\mathbf{E}}[\mathbf{X}_{j-\tau}^{(k)-} | \vec{\mathbf{X}}_j^{(k)-} = \vec{\mathbf{x}}_j^{(k)}] &= \hat{\mathbf{E}}[\mathbf{X}_{j-\tau}^{(k)-} | \mathbf{X}_j^{(1)-} = \mathbf{x}_j^{(1)}] = \\
&\mu_{\mathbf{X}^{(k)-}} + 2 \sum_{\nu=1}^{\infty} \mathbf{Re} \left[ \tilde{\mathbf{f}}_{S_{j-\tau+1}, j} (i2\pi\nu) \left( p^{(k)} e^{-i2\pi\nu u_j^{(1)}} + (1-p^{(k)}) e^{-i2\pi\nu u_j^{(2)}} \right) \times \right. \\
&\quad \left. \int_{S^{(k)}} \cdots \int_{S^{(1)}} \mathbf{x}^{(k)} f_{\vec{\mathbf{X}}^{(k)-}}(\vec{\mathbf{x}}^{(k)}) \left( \xi e^{-i2\pi\nu \xi F_{\mathbf{X}^{(1)-}}(\mathbf{x}^{(1)})} + (1-\xi) e^{i2\pi\nu(1-\xi) F_{\mathbf{X}^{(1)-}}(\mathbf{x}^{(1)})} \right) d\vec{\mathbf{x}}^{(k)} \right]
\end{aligned} \tag{7.16}$$

**Proof.**

Similar to the proof in Proposition 7.2.  $\square$

For a prescribed significance level  $\alpha$ , a  $1 - \alpha$  confidence interval for  $\mathbf{X}_{j+\tau}^{(k)}$  about its point estimator,  $\hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \vec{\mathbf{X}}_j^{(k)}]$ , could be obtained in several ways. One approach is to use equal-width two-sided confidence intervals of the form

$$\left[ \hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \vec{\mathbf{X}}_j^{(k)}] - \gamma, \hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \vec{\mathbf{X}}_j^{(k)}] + \gamma \right] = \left[ \hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \mathbf{X}_j^{(1)}] - \gamma, \hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \mathbf{X}_j^{(1)}] + \gamma \right],$$

where  $\Pr \left\{ \left| \mathbf{X}_{j+\tau}^{(k)} - \hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \vec{\mathbf{X}}_j^{(k)}] \right| > \gamma \right\} = \Pr \left\{ \left| \mathbf{X}_{j+\tau}^{(k)} - \hat{\mathbf{E}}[\mathbf{X}_{j+\tau}^{(k)} | \mathbf{X}_j^{(1)}] \right| > \gamma \right\} = \alpha$ .

Another approach is to use a symmetric confidence interval, about the point estimator. In this paper, we shall adopt the first approach and compute the requisite  $1 - \alpha$  two-sided equal-width confidence interval via a line search utilizing the estimated cdf  $\hat{\mathbf{F}}_{\mathbf{X}_{j+\tau}^{(k)} | \mathbf{X}_j^{(1)}}(\mathbf{y} | \mathbf{x})$ , which can be readily computed from Eq. (7.4).

## 7.2. Selection of the Mixing Parameter

The selection of the mixing parameter takes advantage of time reversal [Kelly (1979), Jagerman and Melamed (1995)]. This approach exploits the fact that the Markov property of background MARM processes is preserved under time reversal, and the transition density of the reversed process can be readily computed, as shown in Section 4.

We can construct estimated time-reversed conditional expectations of the form

$$\hat{\mathbf{E}}[\mathbf{X}_{j-\tau}^{(k)} | \vec{\mathbf{X}}_j^{(k)} = \vec{\mathbf{x}}] = p^{(k)} \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = u_j^{(1)}] + (1-p^{(k)}) \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = u_j^{(2)}]. \tag{7.17}$$

(see Proposition 7.1). To obtain a mixing parameter  $p^{(k)}$  for each dimension  $k$ ,  $1 \leq k \leq N$ , and each time index  $j$ , we minimize the sum of the squared deviations of the estimator of the conditional expectation  $\hat{\mathbf{E}}[\mathbf{X}_{j-\tau}^{(k)} | \vec{\mathbf{X}}_j^{(k)} = \vec{\mathbf{x}}_j^{(k)}] = \hat{\mathbf{E}}[\mathbf{X}_{j-\tau}^{(k)} | \mathbf{X}_j^{(1)} = \hat{\mathbf{x}}_j^{(1)}]$  from the observed history data  $\{\hat{\mathbf{x}}_j^{(k)}\}$ . That is, we minimize the following objective function

$$g_j(p^{(k)}) = \sum_{\tau=1}^T w_{\tau}^{(k)} \left[ p^{(k)} \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = u_j^{(1)}] + (1-p^{(k)}) \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = u_j^{(2)}] - \hat{\mathbf{x}}_{j-\tau}^{(k)} \right]^2 \tag{7.18}$$

where the  $w_{\tau}^{(k)}$  are weights and  $\hat{\mathbf{x}}_{j-\tau}^{(k)}$  is the observed value of  $\mathbf{X}_{j-\tau}^{(k)}$ . The minimizer,  $p^{(k)}$ , of Eq. (7.18) is given by the following proposition.

**Proposition 7.4**

For  $1 \leq k \leq N$ ,  $\tau > 0$  and  $j \geq \tau$ ,

$$\mathbf{p}^{(k)} = \frac{\sum_{\tau=1}^T w_{\tau}^{(k)} \left( \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = \mathbf{u}_j^{(1)}] - \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = \mathbf{u}_j^{(2)}] \right) \left( \hat{\mathbf{x}}_{j-\tau}^{(k)} - \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = \mathbf{u}_j^{(2)}] \right)}{\sum_{\tau=1}^T w_{\tau}^{(k)} \left( \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = \mathbf{u}_j^{(1)}] - \mathbf{E}[\mathbf{X}_{j-\tau}^{(k)} | U_j = \mathbf{u}_j^{(2)}] \right)^2} \quad (7.19)$$

where  $\mathbf{u}_j^{(1)}$  and  $\mathbf{u}_j^{(2)}$  are given by Eq. (7.2).

**Proof.**

Standard minimization by differentiating  $g_j(\mathbf{p}^{(k)})$  with respect to  $\mathbf{p}^{(k)}$ , setting the derivative to zero and then solving for  $\mathbf{p}^{(k)}$ .  $\square$

The minimizer obtained in Eq. (7.19) is a nominal mixing parameter in that the denominator may vanish or the ratio may not always be a legitimate probability value. Thus, the *actual mixing parameter*,  $\hat{\mathbf{p}}^{(k)}$ , is selected as follows:

1. If the denominator of  $\mathbf{p}^{(k)}$  in Eq. (7.19) vanishes, then select any probability value of  $\hat{\mathbf{p}}^{(k)}$ , say,  $\hat{\mathbf{p}}^{(k)} = 1$ .
2. Otherwise, if  $\mathbf{p}^{(k)}$  in Eq. (7.19) is negative, then set  $\hat{\mathbf{p}}^{(k)} = 0$ ; and if it exceeds 1, then set  $\hat{\mathbf{p}}^{(k)} = 1$ .
3. In all other cases, set  $\hat{\mathbf{p}}^{(k)}$  to  $\mathbf{p}^{(k)}$  of Eq. (7.19).

**8. Conclusion**

This paper defined a new class of vector-valued stochastic processes, called MARM. It described the construction of two flavors of MARM processes,  $\text{MARM}^+$  and  $\text{MARM}^-$ , thereby extending previous work on ARM processes (one-dimensional MARM processes) in [Melamed (1999)]. It studied the statistics of MARM processes (transition structure and second order statistics), and devised MARM-based fitting and forecasting algorithms. The MARM fitting methodology produces high-fidelity multivariate models fitted to a prescribed statistical signature by simultaneously fitting first-order statistics (the multi-dimensional marginal distribution) and second-order statistics (autocorrelations and cross-correlations). The MARM forecasting methodology provides both point estimators and confidence intervals.

The key advantage of MARM processes is its ability to fit a strong statistical signature consisting of first-order and second-order statistics simultaneously. More precisely, MARM processes exactly fit an arbitrary multi-dimensional distribution and approximately fit the leading autocorrelation and cross-correlation coefficients. This ability appears to render the MARM modeling methodology unique vis-à-vis the goal of fitting a model to a class of strong statistical signatures.

The class of MARM processes has a number of shortcomings. First, MARM processes are not invariant under permutations of the underlying time series components of the underlying time series vector (abstract or empirical). More specifically, the definition of MARM processes depends on the ordering of individual dimensions in the marginal multi-dimensional distribution in that the ordering affects both fitting and forecasting results. Unfortunately, the construction of MARM processes does not allow us to eliminate this dependency on ordering. However, we may explore and select an ordering that provides the best MARM model among several ordering candidates. Second, MARM modeling suffers from several computational “curse of dimensionality” aspects which limit the MARM fitting algorithm to moderate-size search spaces when running on a contemporary PC. For example, the number of empirical histogram cells grows geometrically in the dimension,  $N$ , while the number of correlation functions grows quadratically in  $N$ . This results in a commensurate increase in computational time as well as the problem of acquiring sufficient data to properly populate a multi-dimensional empirical histogram. Additionally, since the innovation densities in the search space are typically step functions, the model search space must be limited to fairly “crude” innovation densities in view of the fact that the model search space grows exponentially in the number of innovation density steps.

In spite of the computational limitations cited above, practical MARM modeling and forecasting is feasible for a moderate dimension,  $n$ . Accordingly, the theoretical and practical efficacy of the numerical formulas in Sections 4-7 will be exhibited in a companion paper, which will specialize these formulas to the practical case of empirically-based statistical signatures. That paper illustrates the efficacy of the proposed MARM modeling and forecasting methodologies, using a software environment, called *MultiArmLab*, which supports these methodologies and allows the comparison of numerically-computed MARM statistics to their empirical and simulated counterparts.

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